No.

Date

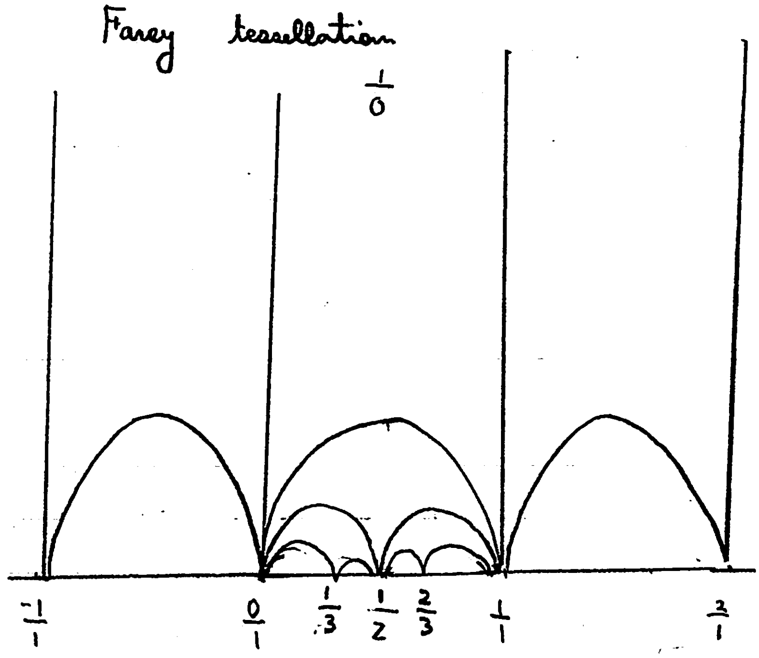
## ④ Examples

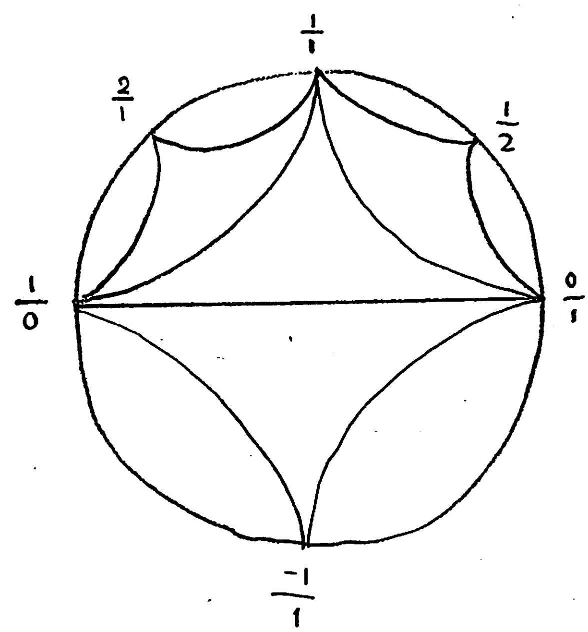
Note on terminology :

It is customary to use " Kleinian groups " to mean discrete subgroups of $\mathrm{Isom}^+ \mathbb{H}^3$.
A discreate subgroup of $\mathrm{Isom}^+ \mathbb{H}^2$ is called a Fuchsian group

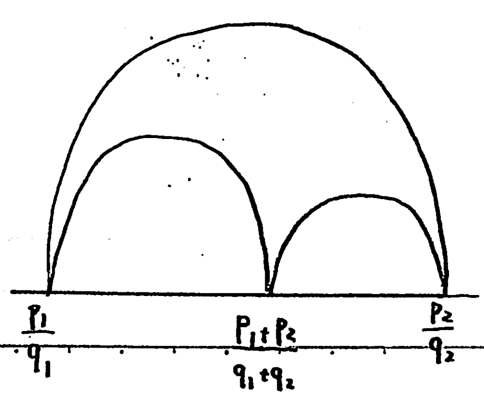
Example 1   $\Gamma = PSL(2, \mathbb{Z}) < PSL(2, \mathbb{R})$.

Farey tessellation



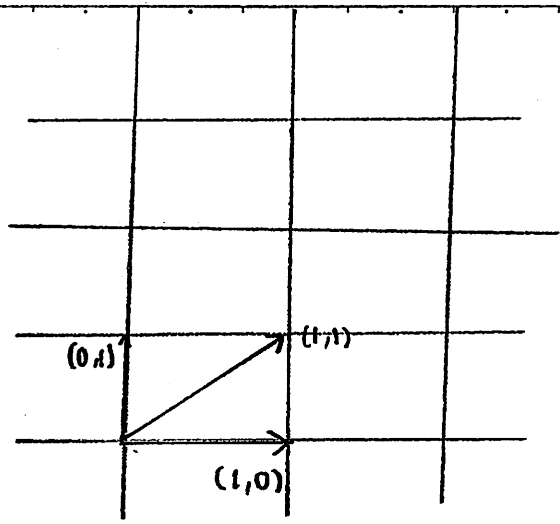




ideal triangle $\langle \frac{0}{1}, \frac{1}{1}, \frac{1}{0} \rangle$
reflection w.r.t. 3 - edges.




$$\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = \pm 1$$

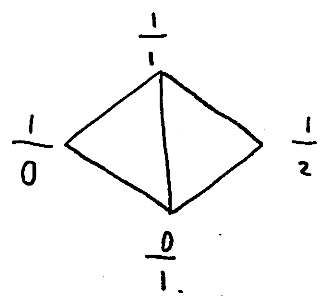
Farey triangle



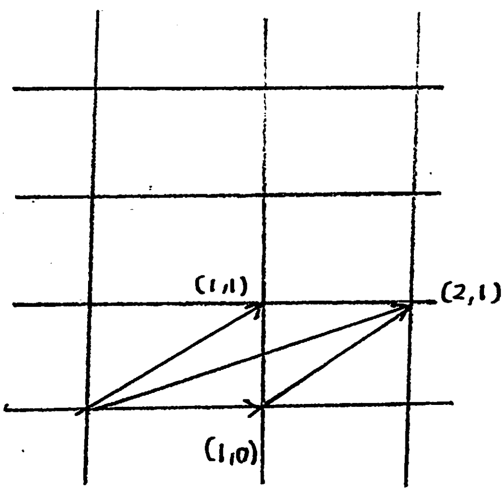

$\langle \frac{0}{1}, \frac{1}{1}, \frac{1}{0} \rangle$

triangulation of $(\mathbb{R}^2, \mathbb{Z}^2)$

$\downarrow$




$\downarrow$




two vectors $(1,0), (1,1)$ are the same :
Replace the other vector with another one
to define a triangulation of $(\mathbb{R}^2, \mathbb{Z}^2)$.

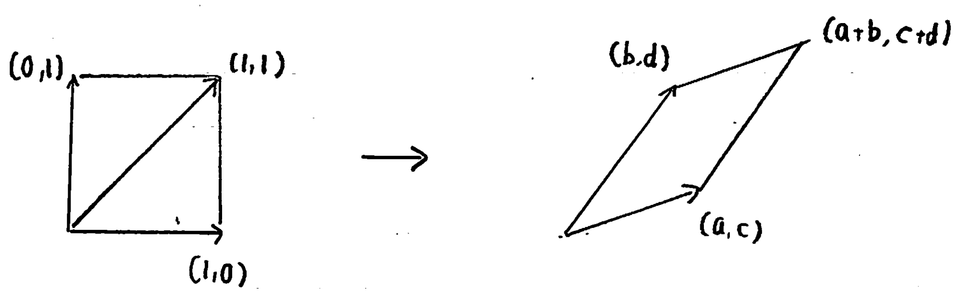
Generally

$$\{\text{Farey triangles}\} \longleftrightarrow \left\{ \begin{array}{l} \text{triangulation of } (\mathbb{R}^2, \mathbb{Z}^2) \\ \text{invariant under translations} \end{array} \right\}$$

No.

Date

$SL(2,\mathbb{Z}) \curvearrowright \{\text{Farey triangles}\}$.




action on the slopes of vectors:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{A}{\mapsto} \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}.$$

The inverse slope of $\begin{pmatrix} x \\ y \end{pmatrix} := \dfrac{x}{y}$.

The inverse slope of $A\begin{pmatrix} x \\ y \end{pmatrix} = \dfrac{ax+by}{cx+dy} = \dfrac{a(x/y)+b}{c(x/y)+d}$

: a linear fractional transf.

$$\frac{0}{1} \mapsto \frac{c}{a}$$

$$\frac{1}{1} \mapsto \frac{c+d}{a+b}$$

$$\frac{0}{1} \mapsto \frac{d}{b}$$

$$A(r) := \frac{ar+d}{cr+d}$$

line of inverse slope $r \overset{A}{\mapsto}$ line of inverse slope $A(r)$

No.

Date

The Vertex set of Farey tessellations

$\hat{\mathbb{Q}} := \mathbb{Q} \cup \{\frac{1}{0}\}$ is identified with

the inverse slopes of lines through $\mathbb{Z}^2$.

Then the action of $A$ on the inverse slopes is the linear fractional transformation

$\Downarrow$

$A$ ( a Farey tessellation ) = a Farey tessellation

The above implies
$PSL(2,\mathbb{Z})$ acts transitively on $\{$ the Farey triangles $\}$.

Question A fundamental domain $D$ of $\overset{\Gamma}{\overset{\shortparallel}{PSL(2,\mathbb{Z})}} \curvearrowright \mathbb{H}^2$. ?

$D \subset$ a Farey triangle $\sigma = \langle 0, 1, \infty \rangle$.

We look at :
$\mathrm{Stab}(\Gamma, \sigma) = \{ A \in \Gamma \mid A(\sigma) = \sigma \}$.

Since $\Gamma$ is orientation-preserving, we see

$A \in \mathrm{Stab}(\Gamma, \sigma)$ satisfies

$$A(0, 1, \infty) = (0, 1, \infty),\ (1, \infty, 0),\ (\infty, 0, 1).$$

Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. Then $\langle A \rangle \subset \mathrm{Stab}(\Gamma, \sigma)$.

Also $A'(0) = 0$, $A'(1) = 1$, $A'(\infty) = \infty$ implies $A' = \mathrm{Id}$. Therefore

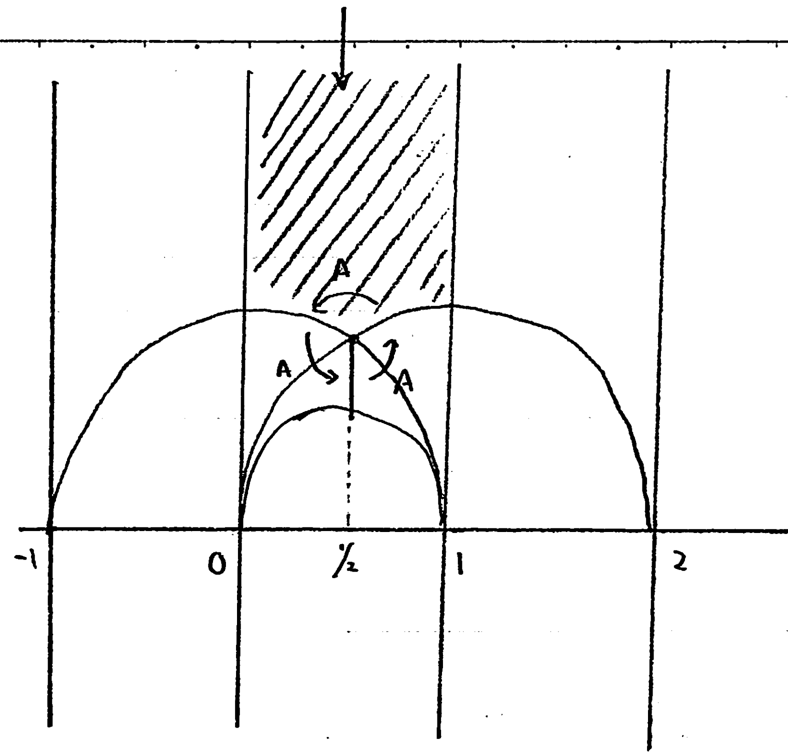
$$\mathrm{Stab}(\Gamma, \sigma) = \langle A \rangle \cong \mathbb{Z}/3\mathbb{Z}.$$

$A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$

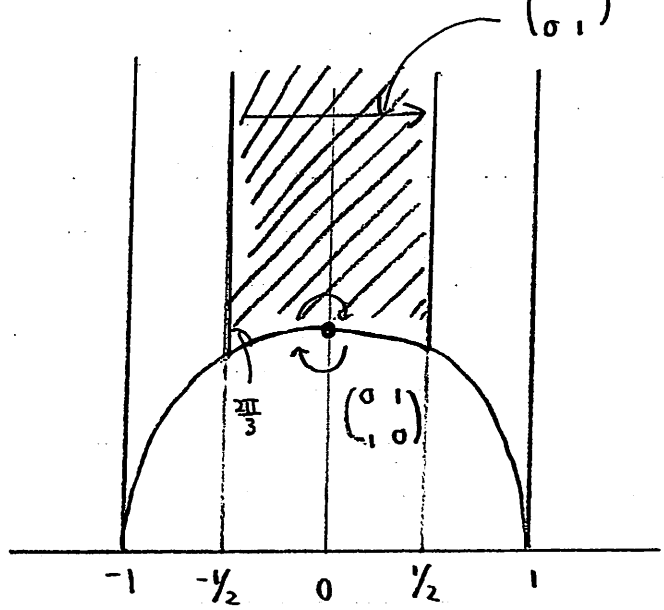
No.

Date

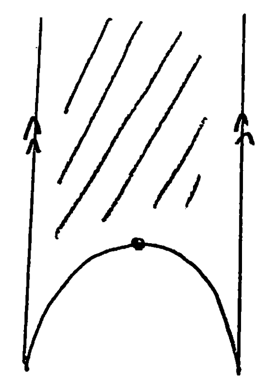
The fundamental domain



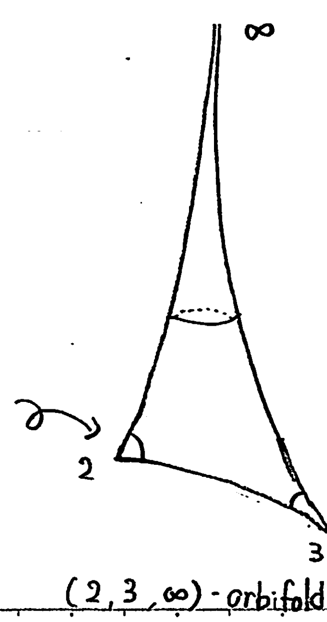

or




$\mathbb{H}^2/\Gamma$ =



≅




modulo $\frac{2\pi}{\infty}$ - "elliptic" transf.

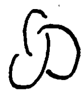
modulo : $\frac{2\pi}{2}$ elliptic transf.

modulo : $\frac{2\pi}{3}$ elliptic trans

(2, 3, ∞) - orbifold

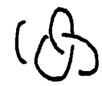
$\mathcal{O} := \mathbb{H}^2/\Gamma$ is a complete hyperbolic orbifold of area $\pi/3$

$\left( \begin{array}{l} (0,1,\infty)\text{-triangle has area } = \pi \\ \text{the fundamental domain } = (0,1,\infty)/3 \\ \quad : \text{area } = \pi/3 \end{array} \right)$

Recall : trefoil knot complement $S^3 -$ 
= a Seifert fibered space / 

Fact : $S^3 -$  $\cong UT\mathbb{H}^2 / PSL(2,\mathbb{Z})$ .

$$\begin{array}{c} UT\mathbb{H}^2 \approx \mathbb{H}^2 \times S^1 . \\ {\scriptstyle S^1}\downarrow \quad \\ \mathbb{H}^2 \quad \end{array}$$

$1 \to \mathbb{Z} \to G($  $) \to \mathbb{Z}/2 * \mathbb{Z}/3 \to 1$ .
$\mathbb{Z}/2 * \mathbb{Z}/3 \cong PSL(2,\mathbb{Z})$ .

In terms of Thurston's terminology, .

trefoil knot complement has $\underbrace{\mathbb{H}^2 \times E^1}_{\widetilde{SL_2(\mathbb{R})}}$ - structure .

No.

Date

Want : Find a manifold w/o singularity.

$\Longleftrightarrow$ find a torsion-free subgroup for Kleinian groups

$A$ is a torsion element $\Longleftrightarrow$ $A$ elliptic

$\Longleftrightarrow$ $\mathrm{Tr}A \in (-2,2) \cap \mathbb{Z} = \{0, \pm 1\}$

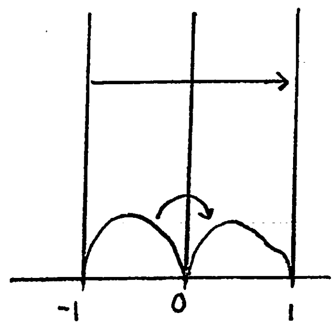
$\Rightarrow$ $A \not\equiv \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2$.

$A \equiv \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2 \Rightarrow A$ : torsion free.

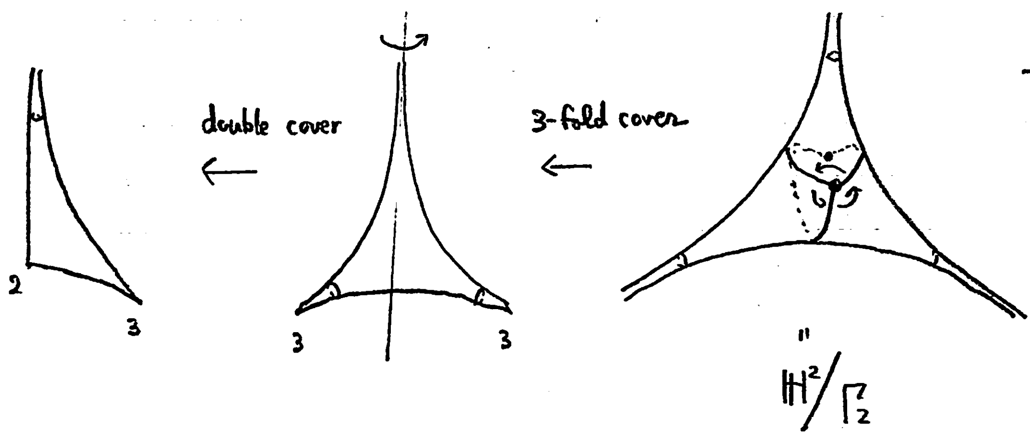
$\Gamma_2 := \{ A \in PSL(2,\mathbb{Z}) \mid A \equiv E_2 \bmod 2 \}$.

$= \mathrm{Ker}\,( PSL(2,\mathbb{Z}) \to PSL(2, \mathbb{Z}/2\mathbb{Z}) )$ ($\because$ torsion-free)

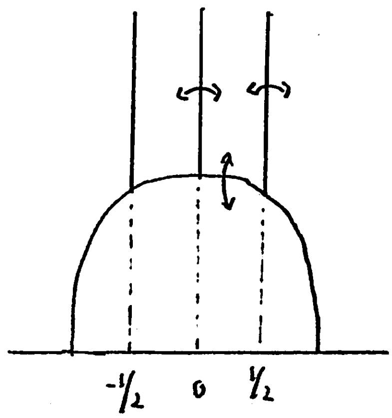
$\cong D_6$ the dihedral group of order 6.




$\mathbb{H}^2/\Gamma_2$ is a hyperbolic surface

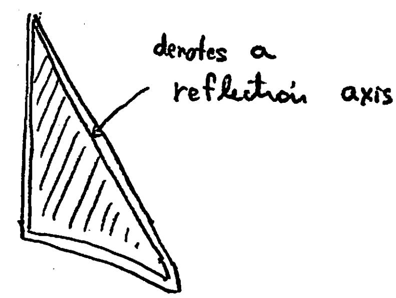


An extension of $PSL(2, \mathbb{Z})$.




an extended triangle group of type $(2, 3, \infty)$
$\|$
the subgroup gen by these three edges.

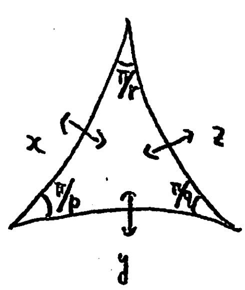
Quotient




$p, q, r \in \mathbb{N}$ with
$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

$\Delta_{p,q,r}$ : a hyperbolic triangle of $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$

$\Delta_{p,q,r}$




The extended triangle group of type $(p, q, r)$.
= the subgroup of $\mathrm{Isom}^{+}(\mathbb{H}^2)$
gen. by $x, y, z$

Theorem.

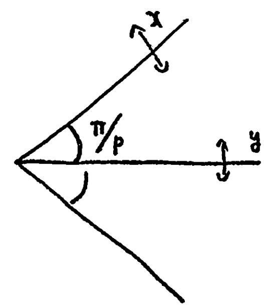
$[p,q,r]$ : a discrete group of $\mathrm{Isom}(\mathbb{H}^2)$ s.t.
the images of $\Delta_{p,q,r}$ under $[p,q,r]$ yields a tessellation of $\mathbb{H}^2$.

$$[p,q,r] = \left\langle x, y, z \;\middle|\; \begin{array}{l} x^2 = y^2 = z^2 = 1 \\ (xy)^p = (yz)^q = (zx)^r = 1 \end{array} \right\rangle$$

Poincaré's fundamental polyhedron theorem

Epstein - Petronio

小島定吉.



$xy = \frac{2\pi}{p}$ rot.

Triangle group $(p,q,r) = [p,q,r] \cap \mathrm{Isom}^+ \mathbb{H}^2$

$$= \langle a, b, c \mid a^p = b^q = c^r = 1, \quad abc = 1 \rangle$$

$$(a = xy, \; b = yz, \; c = zx)$$

Klimenko - Sakuma

$$\mathrm{rank}\,[p,q,r] = \begin{cases} 3 & \\ 2 & \text{if } (p,q,r) = (2,q,r) \quad q \text{ or } r : \text{odd} \\ & \phantom{\text{if } (p,q,r) = } (3,3,r) \quad r \not\equiv 0 \mod 3 \end{cases}$$

No.

Date

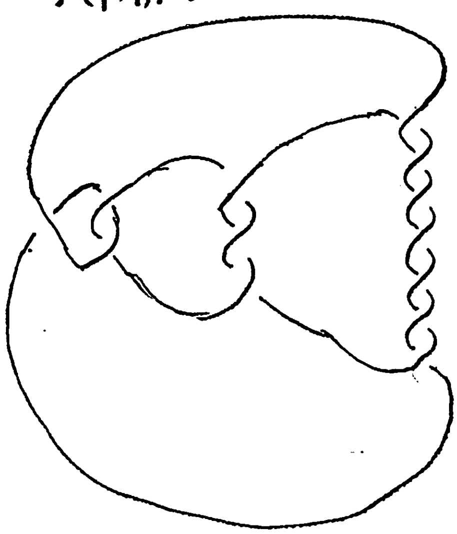
Discreteness Criterion Problem : (open for $n \geq 3$)

$f, g \in \mathrm{Isom}\,\mathbb{H}^n$
Determine whether $\langle f, g \rangle$ is discrete or not.

Remark

(1) $S^3 \setminus K_{p,q} \cong UT\mathbb{H}^2 / (p, q, \infty)$

(2) $UT\mathbb{H}^2 / (p, q, r) \cong$ the double branched cover of Pretzel knot. $P(p, q, r, -1)$



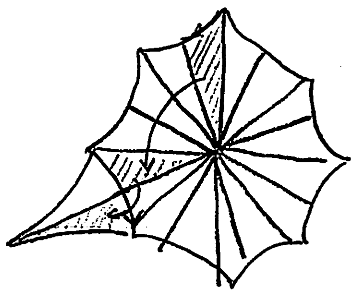
$P(-1, 2, 3, 7) = P(-2, 3, 7)$.

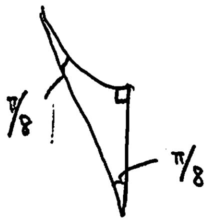
March 2nd week :
Branched coverings,
degenerations and
related topics
Hiroshima

No.

Date

$S_2$






A torsion-free subgroup $\Gamma$ of triangle group $(2, 8, 8)$ produce $\mathbb{H}^2/\Gamma \cong S_2$

$\mathrm{Tech}(S_g) :=$ the complete hyperbolic str$/$isotopy $\cong \mathbb{R}^{6g-6}$

Exe

$S_{g,n}$ = the $n$-punctured surface of genus $g$.

Construct a complete hyperbolic str. on $S_{g,n}$ by means of a triangle group.