No.

Date

③ Hyperbolic Geometry

n-dim'l hyperbolic space : an upper half-model

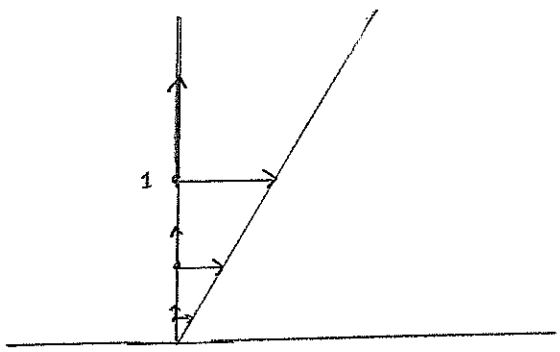
$$\mathbb{H}^n := \{ (x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n > 0 \}$$

with Riemannian metric

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}$$

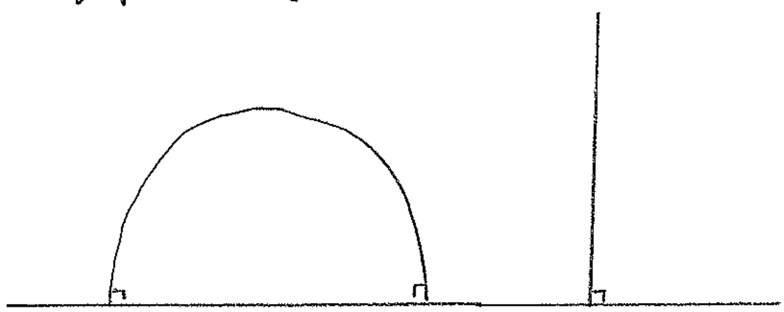
$n = 2$.

o.n.b's of tangent spaces :



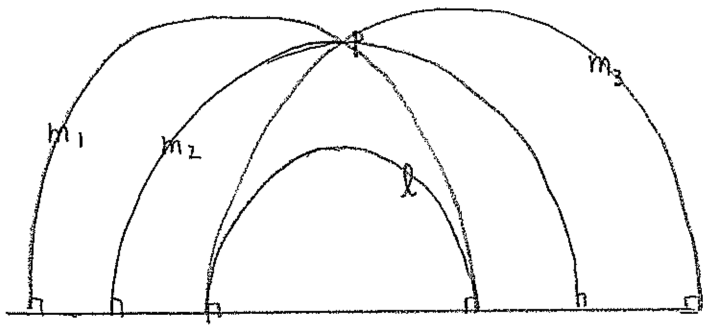
geodesics

the half-sphere orthogonal to $\partial \mathbb{H}^n = (\mathbb{R}^{n-1} \times 0) \cup \{\infty\}$.



$n = 2$

$\exists\infty$ parallel lines through p



$m_i \,/\!/\, l$

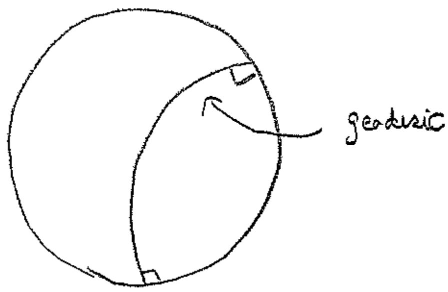
$i = 1, 2, 3, \cdots$

a ball model.

$$\mathbb{B}^n = \{ \overset{x}{\overset{\shortparallel}{(x_1, \cdots, x_n)}} \in \mathbb{R}^n \mid |x| < 1 \} \qquad |x| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}.$$

with R. metric

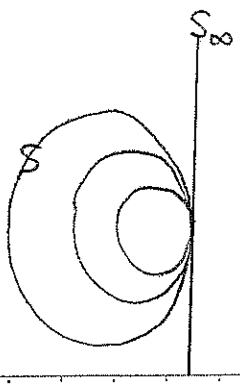
$$ds^2 = \frac{4 \left(\sum_{i=1}^{n} dx_i^2 \right)}{(1 - |x|^2)^2}$$



<u>Exe</u>. Show that $\mathbb{H}^n \underset{\text{isometric}}{\cong} \mathbb{B}^n$. (See the next page)

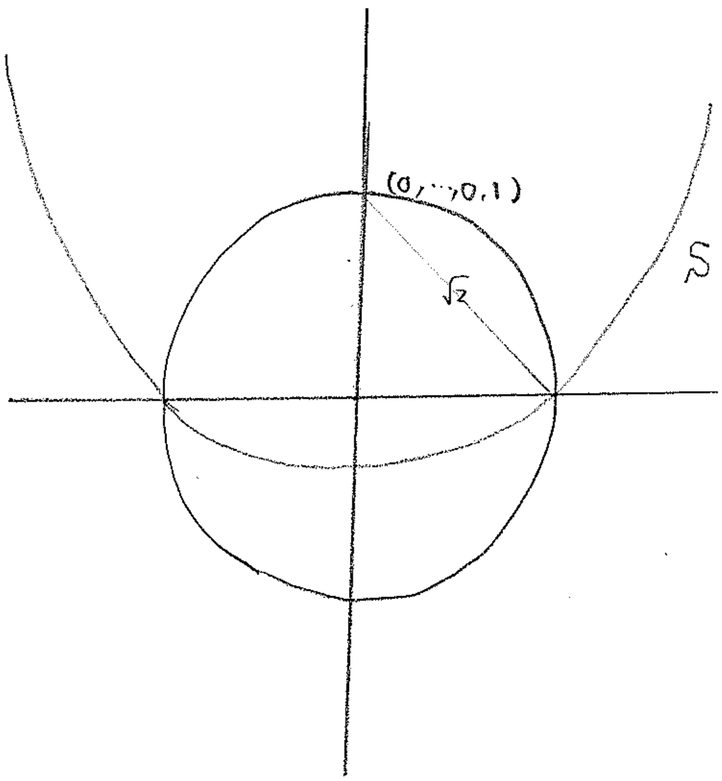
<u>Exe</u> i_S : the inversion w.r.t S.

the radius $S \to \infty \rightsquigarrow i_S \to$ the reflection w.r.t. a hyperplane S_∞



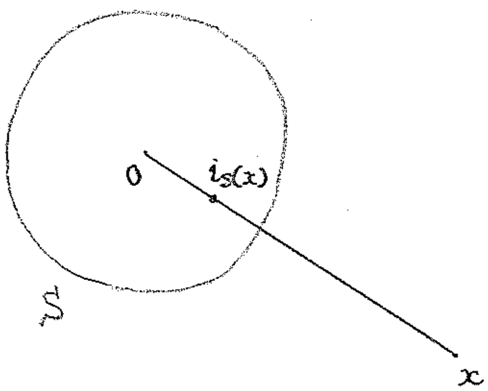
No.

Date



S : $(n-1)$ dim'l sphere of radius $\sqrt{2}$ centered at $(0,\cdot,0,1)$.

$i_S : \mathbb{H}^n \to \mathbb{B}^n$: the inversion wrt S is an isometry



$\overline{O\, i_S(x)} \cdot \overline{Ox} = (\text{the radius of } S)^2$

Isometry group $\quad \mathrm{Isom}\,\mathbb{H}^n$ = the group of all isometries of $\mathbb{H}^n$

∇

$\mathrm{Isom}^+(\mathbb{H}^n)$ = the subgroup of all orientation-pres. isometries of $\mathbb{H}^n$.

Classification of elements of $\mathrm{Isom}^+(\mathbb{H}^n)$

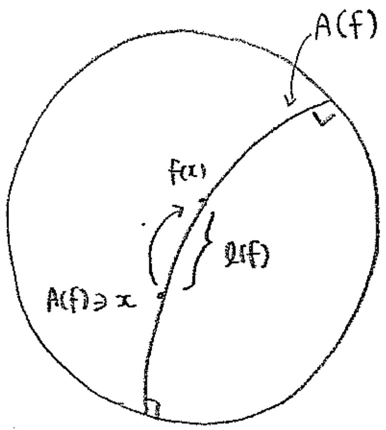
$f \in \mathrm{Isom}^+\mathbb{H}^n$

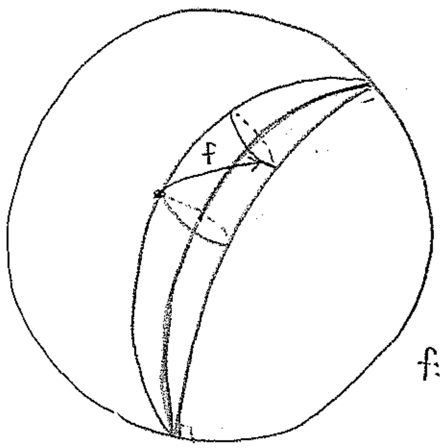
$$\ell(f) := \inf_{x \in \mathbb{H}^n} d(x, f(x)).$$

Case 1 $\quad \ell(f) > 0 \quad \cdots \quad$ loxodromic (or hyperbolic).

$$\ell(f) = \min\{ d(x, f(x)) \mid x \in \mathbb{H}^n \} \quad \&$$

$$A_f := \{ x \in \mathbb{H}^n \mid d(x, f(x)) = \ell(f) \} \text{ is a geodesic in } \mathbb{H}^n.$$





f: a screw-motion around A_f

Transform $\mathbb{B}^n$ to $\mathbb{H}^n$ so that "end points" of $A(f)$ are mapped to $0, \infty$

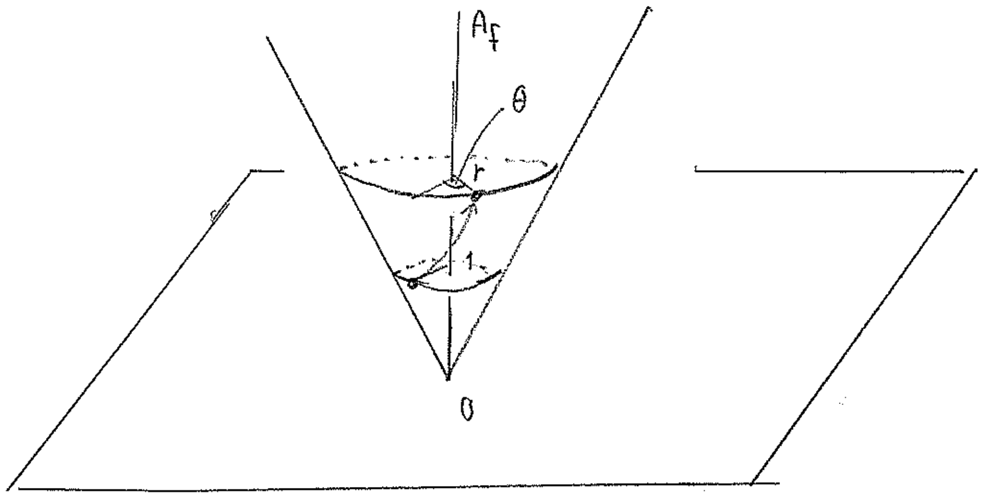
Then the picture looks like, for $n = 3$, as follows

No.

Date

$\mathbb{H}^3 = \{ (z,t) \in \mathbb{C} \times \mathbb{R} \mid t>0 \}$

$f(z,t) = (\lambda z, |\lambda| t)$ $\qquad \lambda = re^{i\theta}$



$l(f) = \log r$ $\qquad$ rotation by θ

The complex translation length of f $= l(f) + i\theta \in \mathbb{C}$

$(l(f) > 0)$

Remark the complex transl. length is defined for each $f \in \mathrm{Isom}^+ \mathbb{H}^n$

f : loxodromic $\equiv$ $l(f) > 0$

$\left(f : \text{(purely) hyperbolic} \equiv \theta = 0 \right)$

Case 2. $\ell(f) = 0$ & $\forall x \in \mathbb{H}^n \quad d(x, f(x)) > 0$.

f : parabolic $\rightsquigarrow$ $\bar{f} : \mathbb{H}^n \cup \partial\mathbb{H}^n \circlearrowleft$ the extension of f.

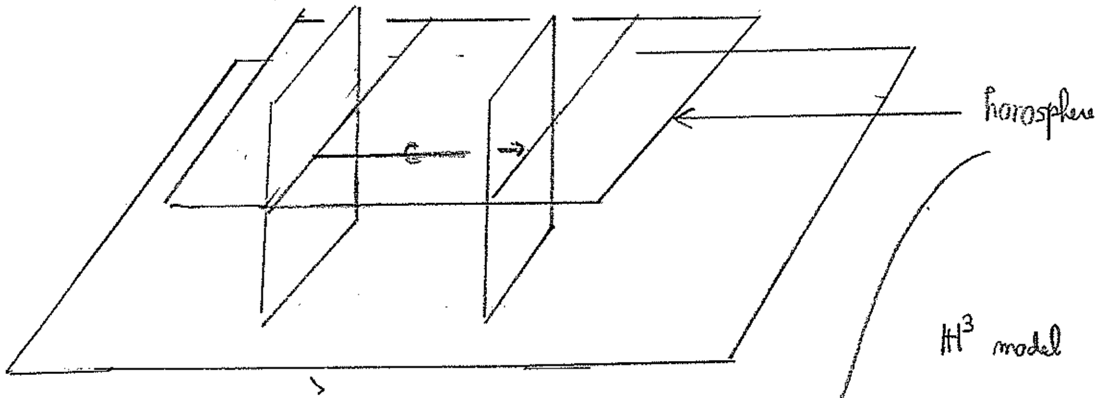
$\exists\, x_i \in \mathbb{H}^n$ s.t. $d(x_i, f(x_i)) \to 0$.

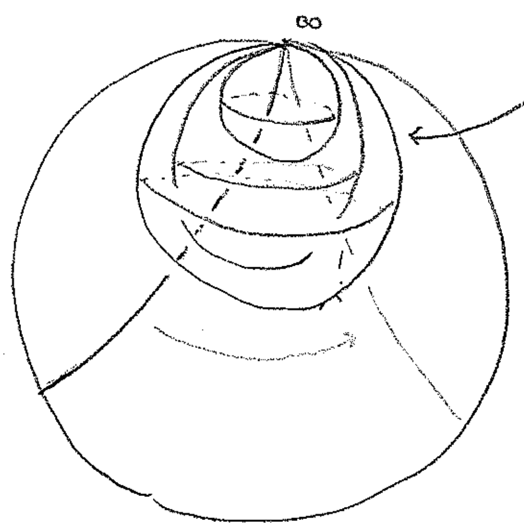
$\Downarrow$

$x_i \longrightarrow x_\infty \in \partial\mathbb{H}^n$. the parabolic fixed pt (uniquely defined).

$n = 3 \qquad x_\infty = \infty$

$f(z,t) = (z + c,\ t)$.





f : parabolic

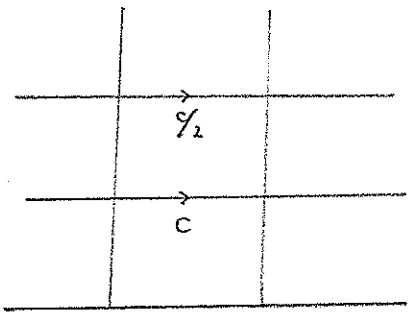
$\Downarrow$

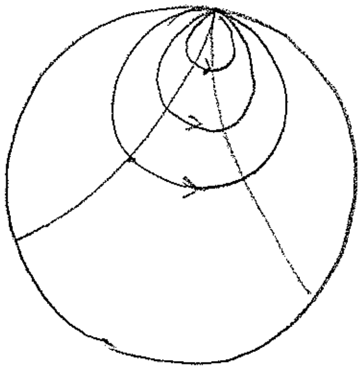
complex translation length $= 0$

No.

Date

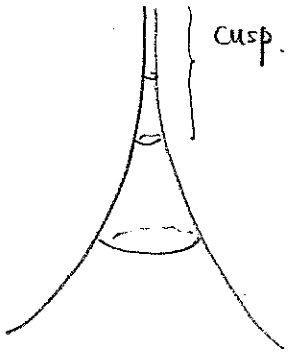
$\mathbb{H}^2$





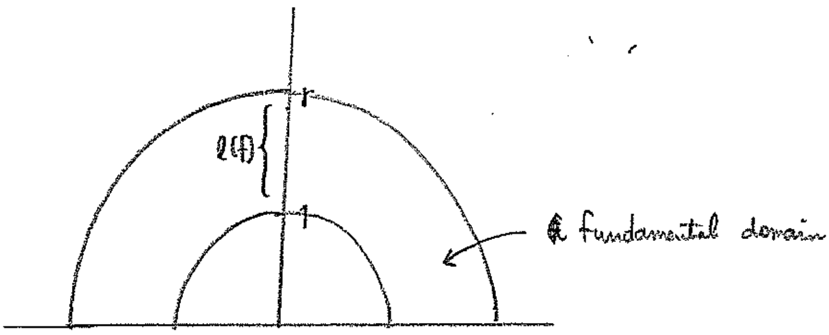
$\mathbb{B}^2$

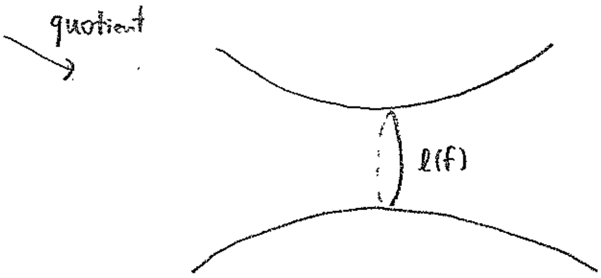
$\mathbb{H}^2/\langle f \rangle$ =



$f : \mathbb{H}^2 \to \mathbb{H}^2$ hyperbolic We describe $\mathbb{H}^2/\langle f \rangle$.

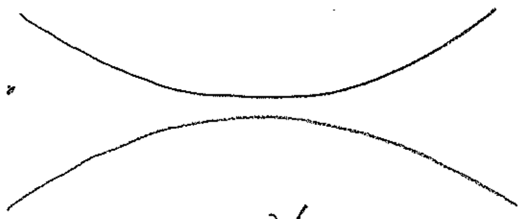
$\mathbb{H}^2 = \{ z \in \mathbb{C} \mid \mathrm{Im}\, z > 0 \}$, $f(z) = rz$ (we assume $r > 1$)





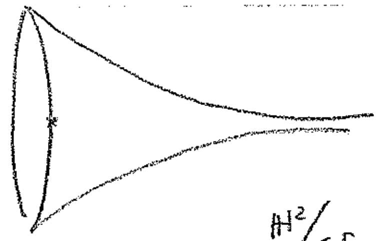
$$\frac{\lambda z + 1}{1}$$

$l(f_n) \to 0 \quad \rightsquigarrow \quad f_n \to f_\infty$
parabolic



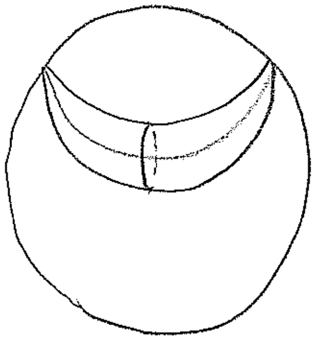
$\mathbb{H}^2/(f_n)$

$\xrightarrow{\text{pointed GH}}$

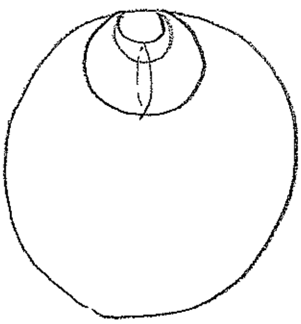


$\mathbb{H}^2/\langle f_\infty \rangle$

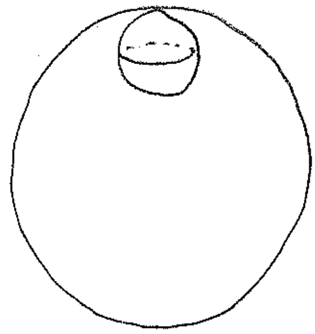
$f_n \to f_\infty$. parabolic
loxodromic



$\to$



$\to$



example

$$\begin{pmatrix} \lambda & 1 \\ 0 & 1/\lambda \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$\lambda \to 1$.

No.

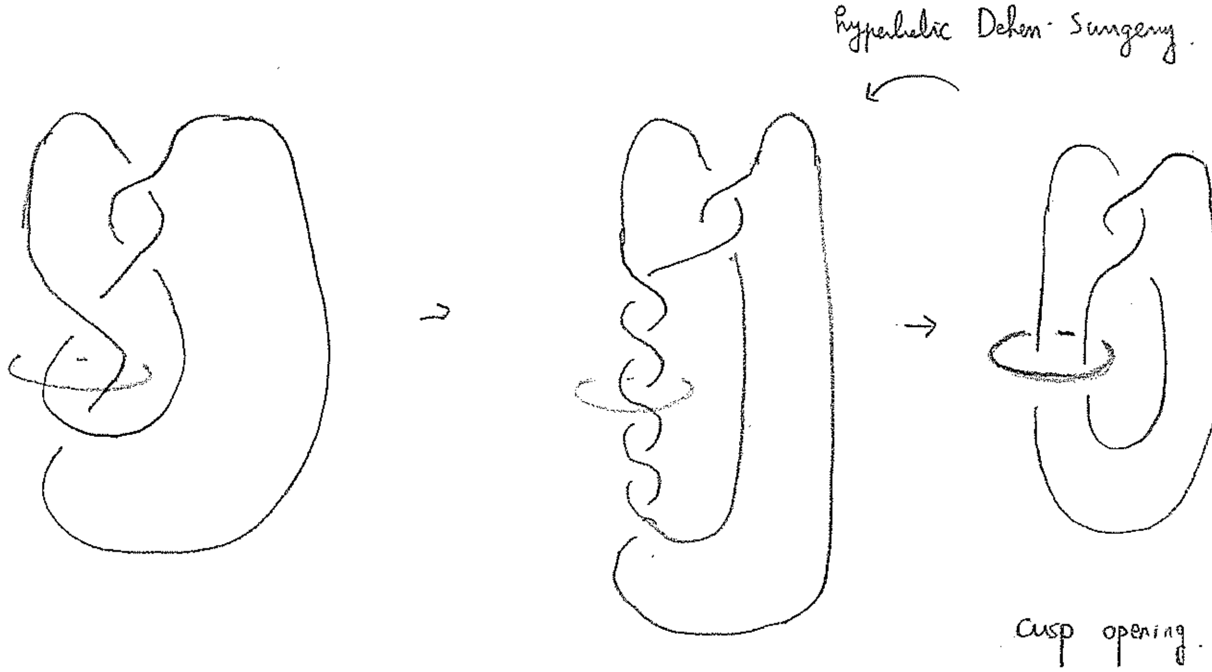
Date

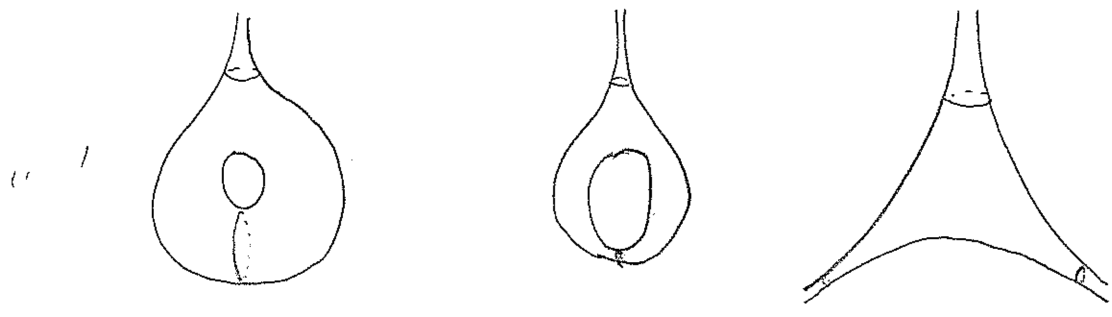
Remark

loxodromic $\xrightarrow[\text{appropriate limit}]{}$ parabolic or, parabolics perturbed to loxodromics.

On the basis of this phenomena,

hyperbolic 3-manifold $\left\{ \begin{array}{l} \text{cusp opening} \\ \text{hyperbolic Dehn surgery} \end{array} \right.$





cf Boileau - Porti

Geometrization of 3-dim'l orbifold Asterisque.

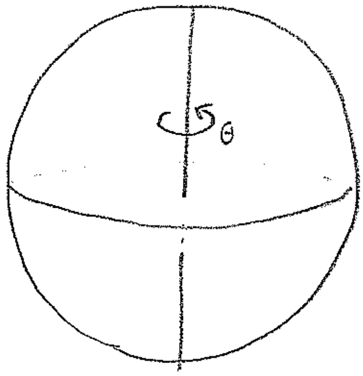
No.

Date . .

Case 3. (for $n=3$)

$\ell(f)=0$ $\quad \mathrm{Fix}\, f \neq \phi$.

A_f = geodesic
$\parallel$
$\mathrm{Fix}\, f$.



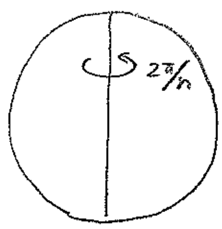
complex translation length $= 0 + i\theta$

$\mathbb{H}^3/\langle f\rangle$

$\frac{\theta}{2\pi} \notin \mathbb{Q} \Rightarrow \mathbb{H}^3/\langle f\rangle$ non-Hausdorff.

$\frac{\theta}{2\pi} \in \mathbb{Q} \Rightarrow \mathbb{H}^3/\langle f\rangle$

order $f = n$



hyperbolic orbifold with cone axis of cone angle $2\pi/n$.

No.

Date

$n = 2$ $\quad \mathrm{Isom}^+ \mathbb{H}^2 \cong PSL(2,\mathbb{R})$.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad f_A(z) = \frac{az+b}{cz+d}$$

<u>Exe</u> Prove $PSL(2,\mathbb{R}) \subset \mathrm{Isom}^+ \mathbb{H}^2$

$n = 3$ $\quad \mathrm{Isom}^+ \mathbb{H}^3 \cong PSL(2,\mathbb{C})$.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad f_A : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$

$$f_A(z) = \frac{az+b}{cz+d}.$$

Viewing $\hat{\mathbb{C}}$ as $\partial \mathbb{H}^3$, we obtain a unique isometric extension $\overline{f_A} : \mathbb{H}^3 \to \mathbb{H}^3$.

<u>Example</u> $A = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$. $\qquad A(z) = \lambda^2 z$

$$\begin{array}{ccc} f_A : \mathbb{H}^3 & \to & \mathbb{H}^3 \\ \Downarrow & & \Downarrow \\ (z,t) & \mapsto & (\lambda^2 z, |\lambda|^2 t) \end{array}$$

complex translation length $= 2 \log \lambda \in \mathbb{C}/2\pi i \mathbb{Z}$.

No.

Date

Prop. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{C})$ $\quad A \neq \pm E$ $\qquad$ $\mathrm{Tr}A$ defined up to sign.

A: loxodromic $\iff$ $\mathrm{Tr}A \notin [-2, 2]$.

parabolic $\iff$ $\mathrm{Tr}A \in \{\pm 2\}$

elliptic $\iff$ $\mathrm{Tr}A \in (-2, 2)$.

<u>proof</u> Examine $f_A : \hat{\mathbb{C}} \circlearrowleft$ $\quad \mathrm{Fix}\, f_A$:

$$z = \frac{az+b}{cz+d} \iff cz^2 + (d-a)z - b = 0$$

$$D = (a+d)^2 - 4(ad-bc)$$
$$= (\mathrm{Tr}A)^2 - 4.$$

case 1. $\mathrm{Tr}A \neq \pm 2$ $\qquad D \neq 0$.

$f_A : \hat{\mathbb{C}} \circlearrowleft$ has two fixed pts. (and A is diagonizalable)

Transform A so that
$\mathrm{Fix}\, f_A = \{0, \infty\}$ $\quad \left(\iff A = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \right)$

$$\bar{f}_A(z, t) = (\lambda^2 z, |\lambda|^2 t) = (r^2 e^{2i\theta} z, r^2 t). \qquad \lambda = re^{i\theta}$$

$\bar{f}_A$: loxodromic $\iff$ $r \neq 1$

elliptic $\iff$ $r = 1$.

$\mathrm{tr}A = re^{i\theta} + \frac{1}{r}e^{-i\theta}$ $\qquad$ Hence $\mathrm{tr}A \in [-2, 2] \iff r = 1$.

Noticing $re^{i\theta} \neq \pm 1$, we obtain the conclusion for case 1.

Case 2. $\mathrm{Tr}A = \pm 2$ $\qquad D = 0$.

$A \neq E$ $\qquad A \underset{\text{conjugate}}{\sim} \pm \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$: parabolic.

$(c \neq 0)$

No.

Date

Def. $\Gamma < \mathrm{Isom}^+ \mathbb{H}^n$ is a Kleinian group. $\equiv$ Γ is discrete

$\Longleftrightarrow$ $\Gamma \curvearrowright \mathbb{H}^n$ is properly discontinuous

i.e. $\forall C$ cpt $\subset \mathbb{H}^n$

$\#\{\gamma \in \Gamma \mid \gamma C \cap C \neq \emptyset\} < \infty$

$\Leftarrow$) easy

$\Rightarrow$) $\forall x \in \mathbb{H}^n$ $\mathrm{Stab}(x, \mathrm{Isom}^+ \mathbb{H}^n) = \{ f \in \mathrm{Isom}^+ \mathbb{H}^n \mid f(x) = x \}$

$\cong SO(n)$ compact

Remark Γ Kleinian group $\not\Rightarrow$ $\Gamma \curvearrowright \underset{\underset{S^{n-1}\ \text{compact}}{\cong}}{\partial \mathbb{H}^n}$! properly discontinuous.

(1) In fact, if $\#\Gamma = \infty$, then the action is NOT properly discontinuous.

(2) $x \in \partial \mathbb{H}^n$

$\mathrm{Stab}(x, \mathrm{Isom}^+ \mathbb{H}^n)$ non-compact.

Limit set of Γ : Kleinian

$x \in \mathbb{H}^n$ fixed

$\Gamma x \subset \mathbb{H}^n$ discrete and has accumulation pts in $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial \mathbb{H}^n$

$\Lambda(\Gamma) :=$ the set of all accumulation pts of Γx. $\subset \partial \mathbb{H}^n$

Γ-inv. closed

No.

Date

$\Omega(\Gamma) := \partial \mathbb{H}^n \setminus \Lambda(\Gamma)$: the domain of discontinuity

Fact $\Gamma \curvearrowright \Omega(\Gamma)$ is properly discontinuous

Remark $\Lambda(\Gamma)$ does not depend on the choice of x.

cf.
Mumford - Series - Wright Indra's pearls.