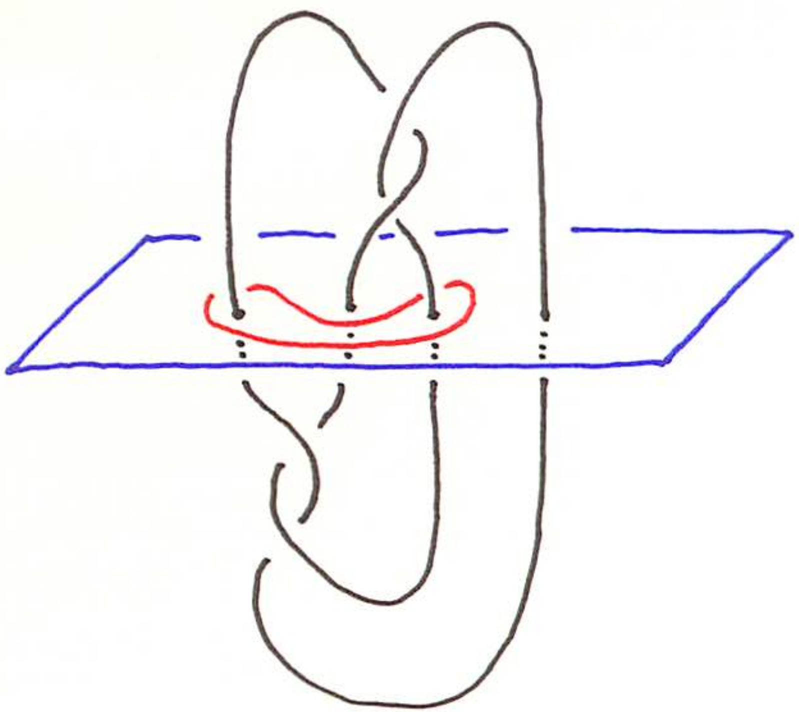
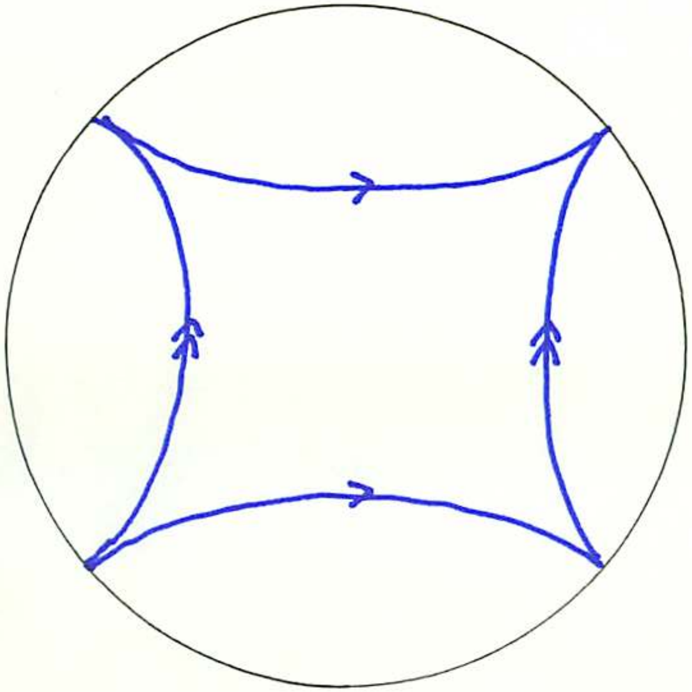
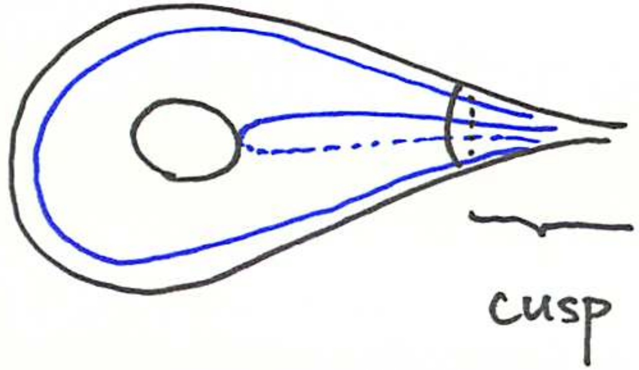
A Variation of McShane's Identity for 2-bridge Links



李 東姫 (釜山)
作間 誠 (広島)

T : once-punctured torus with a complete hyperbolic metric



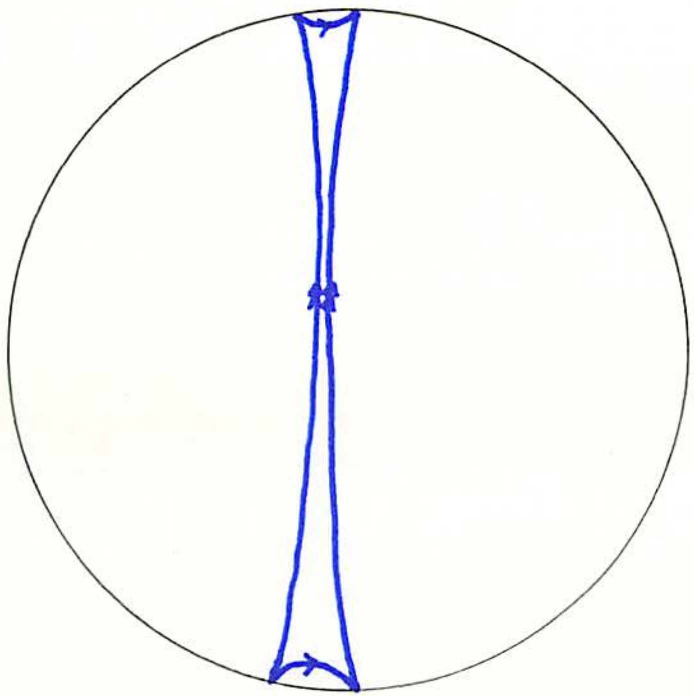


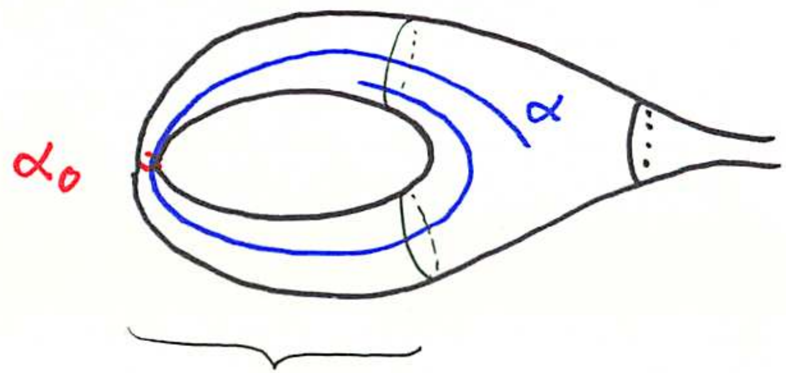
[McShane's identity]

$$\sum_{\alpha} \frac{1}{1+\exp(\ell(\alpha))} = \frac{1}{2}$$

where α runs over simple closed geodesics

[Jorgensen's interpretation]



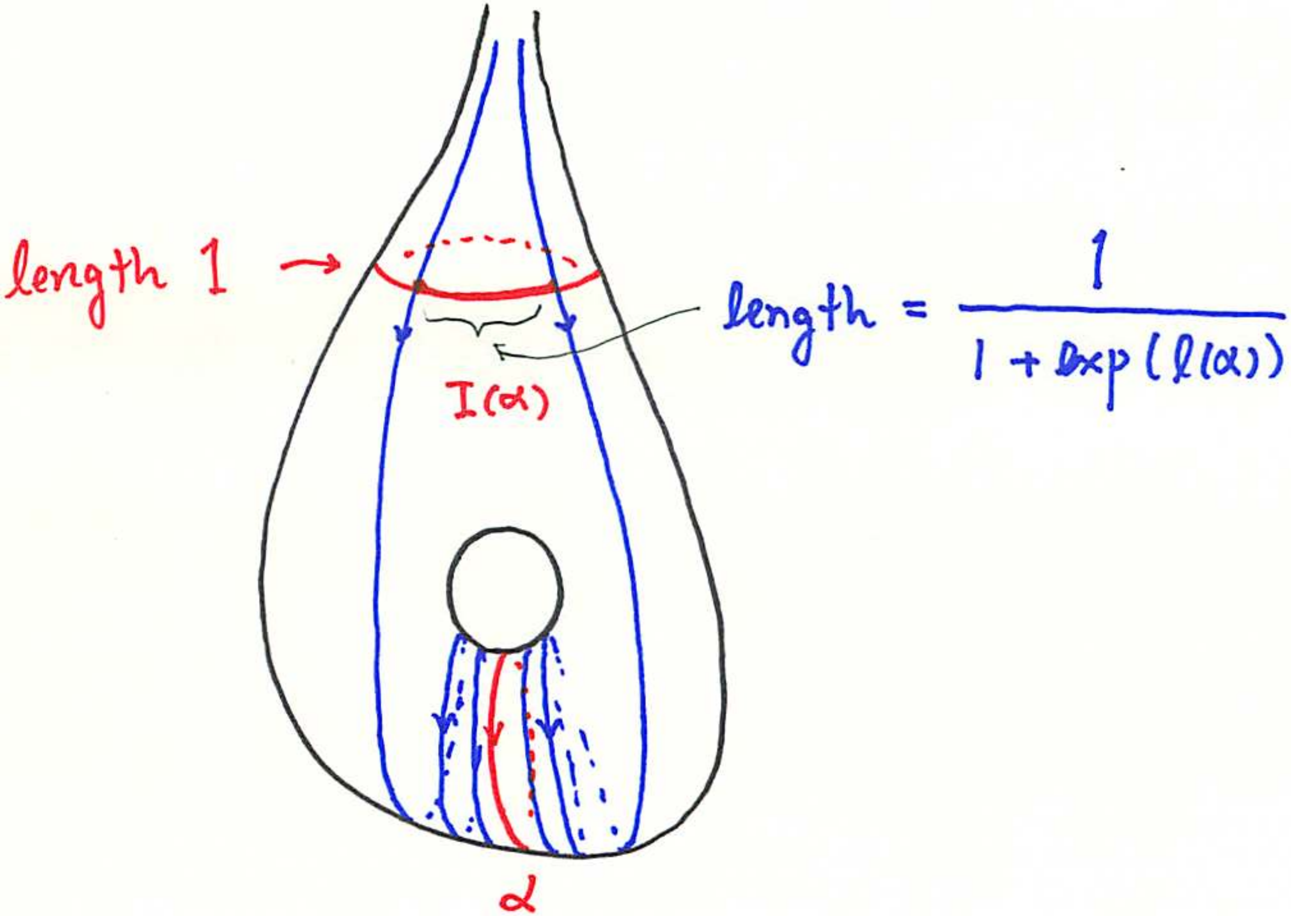


long thin tube

$$\ell(\alpha) \approx \begin{cases} 0 & \text{if } \alpha = \alpha_0 \\ \infty & \text{if } \alpha \neq \alpha_0 \end{cases}$$

$$\frac{1}{1+\exp(\ell(\alpha))} \approx \begin{cases} \dfrac{1}{1+e^0} = \dfrac{1}{2} & \text{if } \alpha = \alpha_0 \\ \dfrac{1}{1+e^\infty} = 0 & \text{if } \alpha \neq \alpha_0 \end{cases}$$

Geometric meaning of the summands

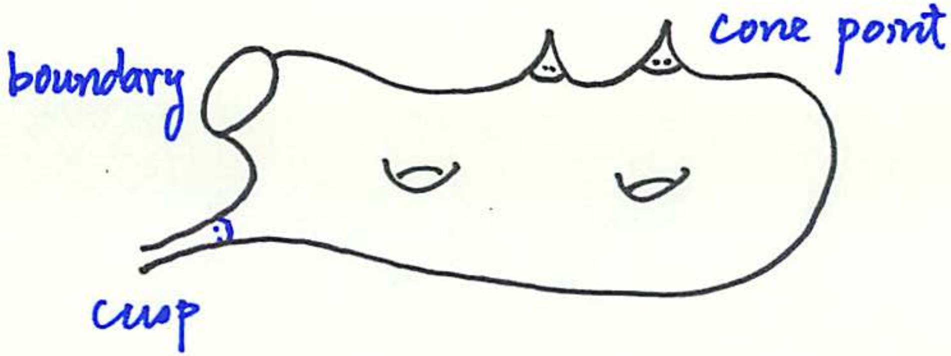


$$(\text{The cusp cross cross section}) - (\text{measure } 0 \text{ Cantor set}) = \bigsqcup_{\alpha} I(\alpha)$$

α: oriented simple geod.

McShane, Tan-Wong-Zhang, Mirzakhani

Generalization to higher genus surfaces
with punctures, boundaries, or cone points



Mirzakhani

Application to Weil-Petersson volumes of moduli spaces
of bordered / cusped hyperbolic surfaces

Bowditch, Akiyoshi-Miyachi-S, Tan-Wong-Zhang

Variations for 3-dim hyperbolic manifolds

$K \subset S^3$ hyperbolic knot

$\Leftrightarrow$ $S^3 - K$ admits a complete hyperbolic structure of finite volume

$\Leftrightarrow$ $S^3 - K \cong \mathbb{H}^3 / \Gamma$ $\quad \Gamma < \mathrm{Isom}^+ \mathbb{H}^3$ discrete, torsion free

$\mathrm{vol}(\mathbb{H}^3/\Gamma) < +\infty$

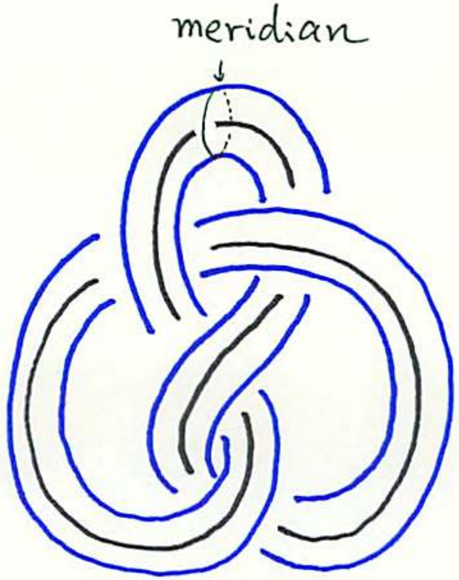
Remark (1) By Mostow's rigidity theorem, the complete hyperbolic structure is unique.

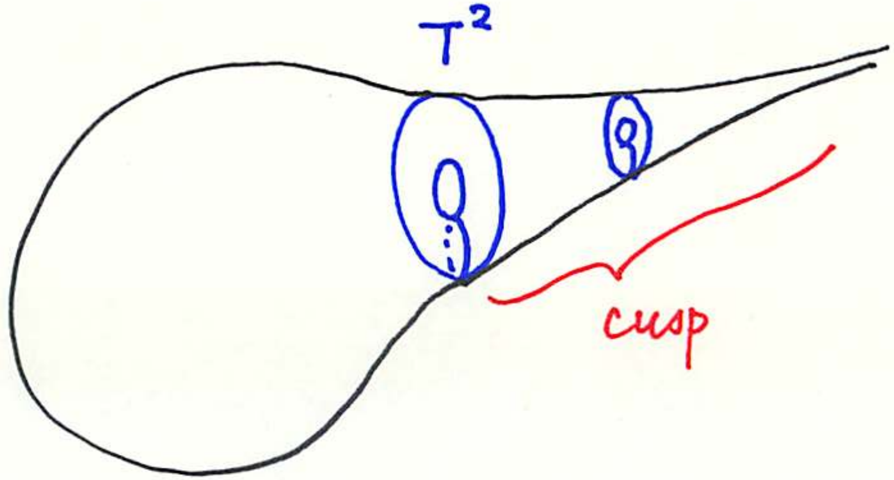
(2) Holonomy representation

$$\rho : G(K) = \pi_1(S^3 - K) \underset{\cong}{\longrightarrow} \Gamma < \mathrm{Isom}^+ \mathbb{H}^3 = PSL(2, \mathbb{C})$$

is faithful, discrete.

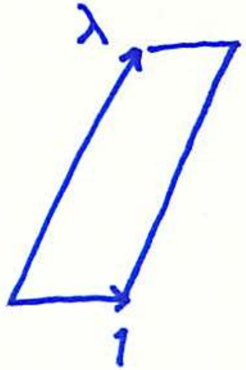
The cusp torus admits a Euclidean structure





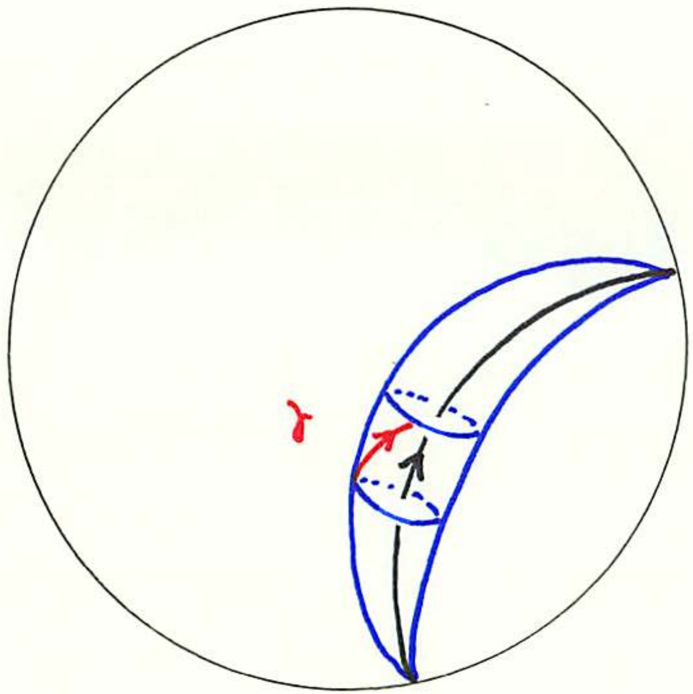
$T^2 \cong \mathbb{C}/\langle 1, \lambda \rangle$ where $1 \leftrightarrow$ meridian

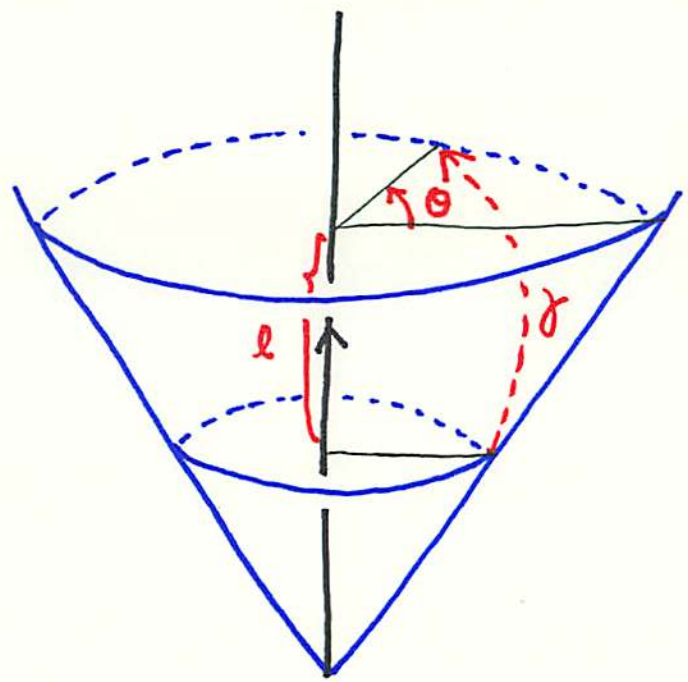
$\lambda \leftrightarrow$ longitude



λ is called the modulus of the cusp torus

Complex translation length of $\gamma \in \mathrm{Isom}^+ \mathbb{H}^3$





Complex translation length $= \ell + i\theta \in \mathbb{C}/2\pi i\mathbb{Z}$

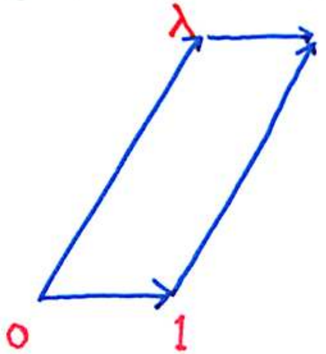
Theorem (A variation of McShane's identity for 2-bridge knots)

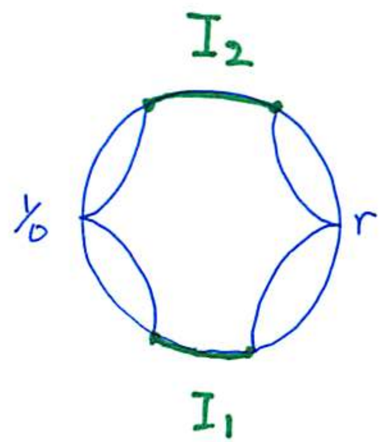
For $\rho_r : \pi_1(T) \to PSL(2, \mathbb{C})$ corresponding to the complete hyperbolic structure of $S^3 - K(r)$;

$$2 \sum_{s \in \mathring{I}_1} \frac{1}{1 + \exp(\ell(\rho_r(\alpha_s)))} + \sum_{s \in \partial I_1} \frac{1}{1 + \exp(\ell(\rho_r(\alpha_s)))}$$

$$= -1 - 2 \sum_{s \in \mathring{I}_2} \frac{1}{1 + \exp(\ell(\rho_r(\alpha_s)))} - \sum_{s \in \partial I_2} \frac{1}{1 + \exp(\ell(\rho_r(\alpha_s)))}$$

= Modulus of the cusp torus with a suitable choice of the longitude.





□

(Outline of proof)

(1) The series converges absolutely.

- A refinement of Bowditch's work by Akiyoshi-Miyachi-S and Tan-Wong-Zhang.
- Discreteness of the length spectrum of finite volume hyperbolic manifolds

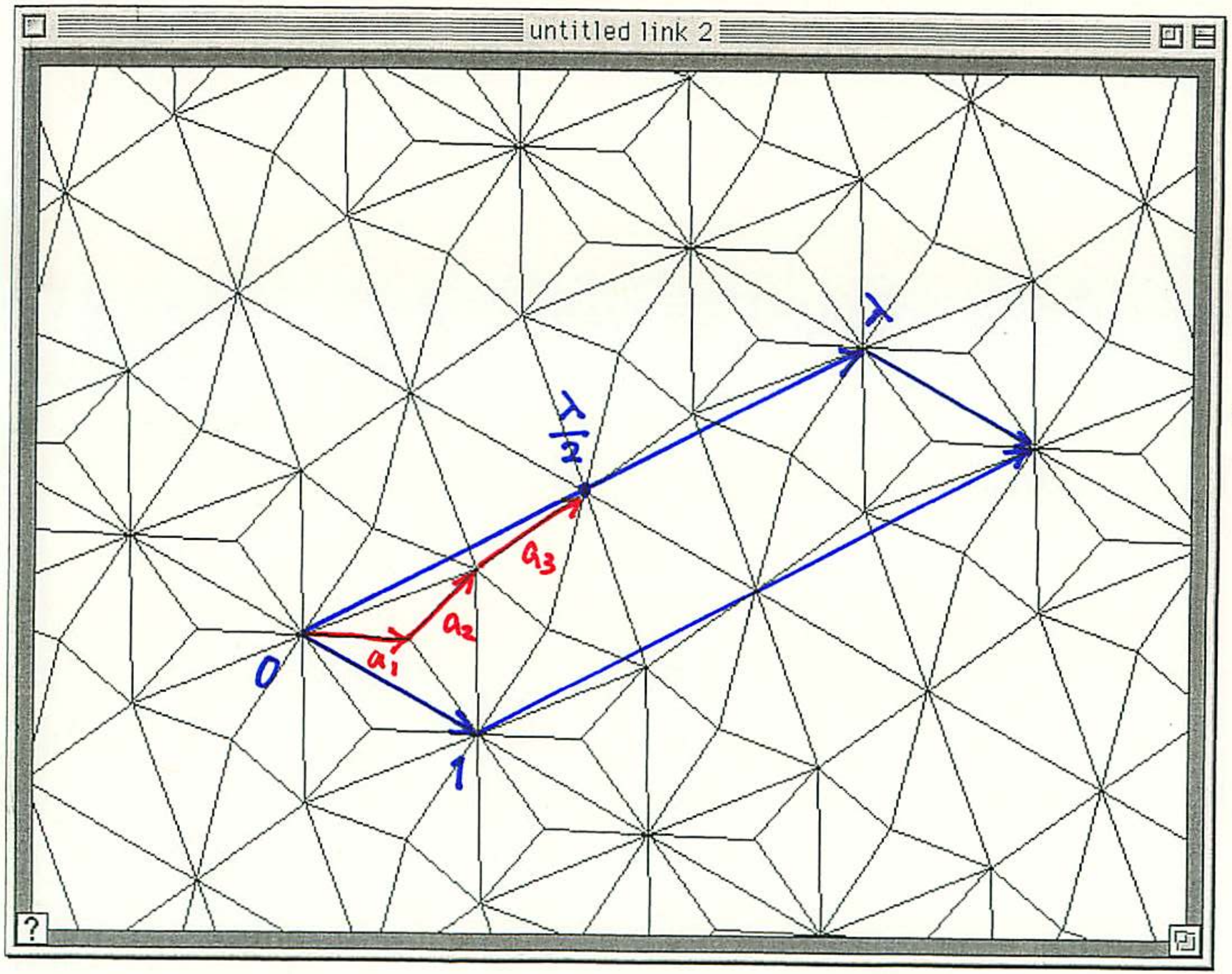
⊙ Explicit solution of some special word problem, conjugacy problem, and peripheral problem for 2-bridge knot groups

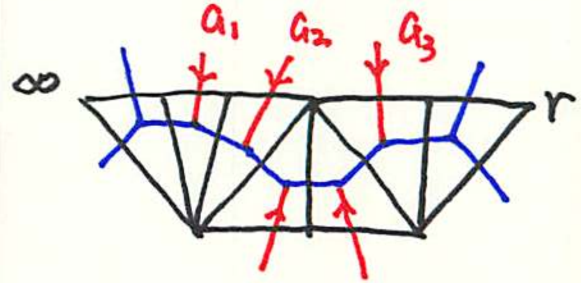
(2) The series represents the modulus of the cusp torus

S-Weeks decomposition of 2-bridge link complements, which are shown to be geometric by [Futer-Gueritaud] [ASWY] □

Cusp triangulation induced by S-W decomposition

$$\frac{\lambda}{2} = a_1 + a_2 + a_3$$





K : knot or link in S^3

S : punctured sphere in $S^3 - K$ obtained from a bridge sphere

Question

(1) For an essential simple loop in S,

when is it null-homotopic in $S^3 - K$?

: : peripheral : :

: : imprimitive : :

(2) For two essential simple loops in S,

when are they homotopic in $S^3 - K$?

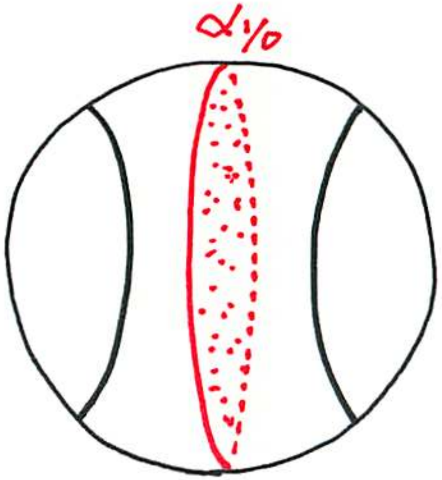
[Lee-S] Complete answers to the above questions

for 2-bridge knots and links

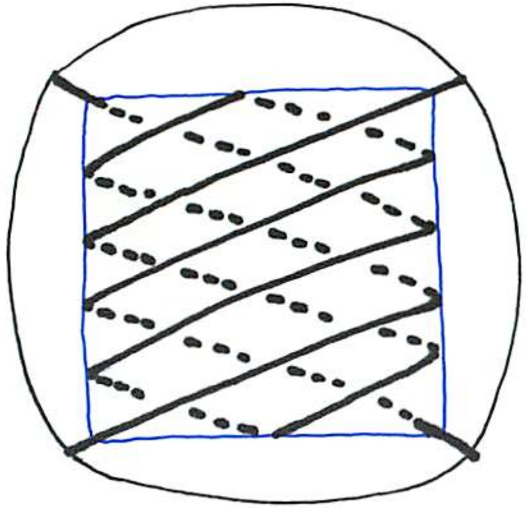
See arXiv 1004.2571, 1010.2232, 1103.0856, 1111.3562

(Proc. London Math Soc) for exposition 1104.3462 (Electric Res. Ann. in Math Sci. 18 (2011))

Rational tangle $(B^3, t(r))$ of slope r :



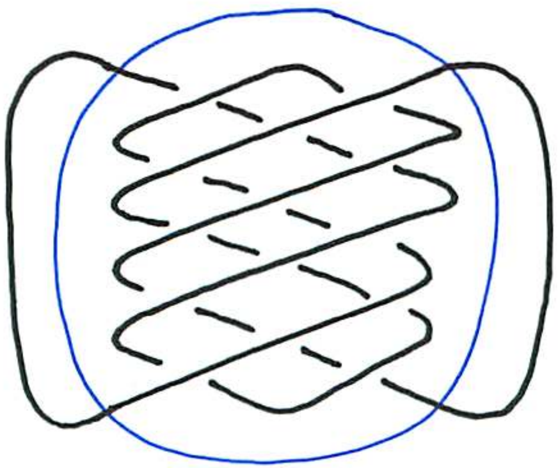
$(B^3, t(1/0))$



$(B^3, t(2/5))$

$$\pi_1(B^3 - t(r)) \cong \pi_1(S) / \langle\langle \alpha_r \rangle\rangle$$

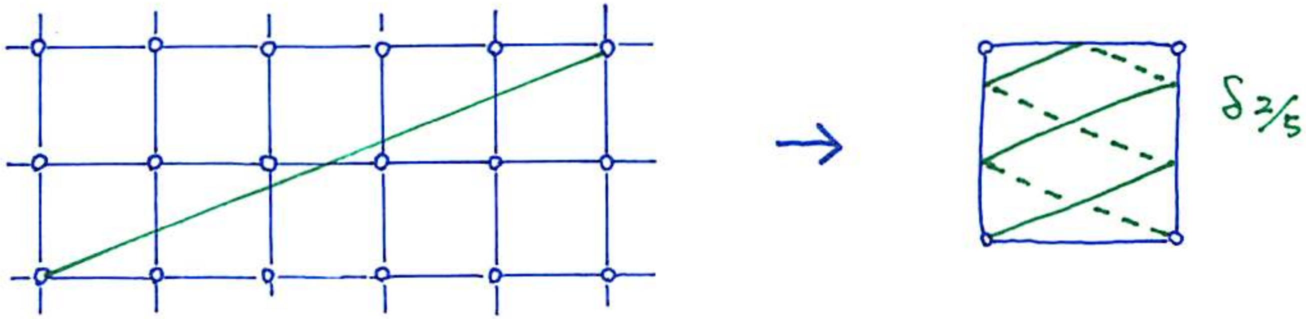
$(S^3, K(r)) = (B^3, t(\infty)) \cup (B^3, t(r))$: 2-bridge link of slope r



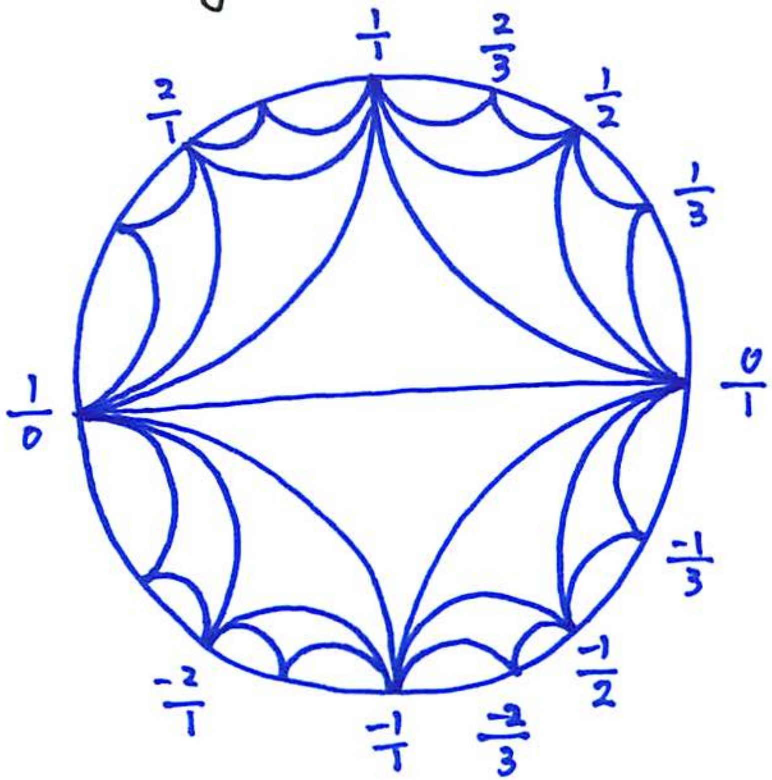
$$G(K(r)) := \pi_1(S^3 - K(r)) \cong \pi_1(S) / \langle\langle \alpha_\infty, \alpha_r \rangle\rangle$$

$S := \mathbb{R}^2 - \mathbb{Z}^2 / \langle \pi\text{-rotations around punctures} \rangle$: 4-punctured sphere

(Conway sphere)



D : Farey tessellation



Vertex set of D $= \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\frac{1}{0}\} \ni r$

$\underset{1-1}{\leftrightarrow}$ {essential simple loops on S} $\ni \alpha_r$

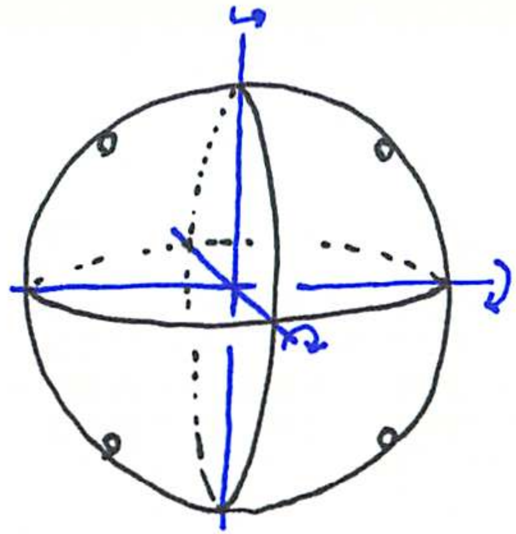
$\underset{1-2}{\leftrightarrow}$ {essential simple arcs on S} $\ni \delta_r$

Farey triangle ideal triangulation of S

Mapping class group $\mathcal{M}(S) := \pi_0 \operatorname{Diff}(S)$

$$1 \to (\mathbb{Z}/2\mathbb{Z})^2 \to \mathcal{M}(S) \xrightarrow{\Phi} \operatorname{Aut}(D) \to 1$$
$$\operatorname{Aut}(D) \cong PGL(2,\mathbb{Z})$$

$(\mathbb{Z}/2\mathbb{Z})^2$-action on S acts trivially on D.



$$\mathcal{M}(S) \overset{\Phi}{\twoheadrightarrow} \mathrm{Aut}(D)$$

$$\cup$$

$$\mathcal{M}(B^3, t(\infty)) := \pi_0 \mathrm{Diff}(B^3, t(\infty))$$

$$\cup$$

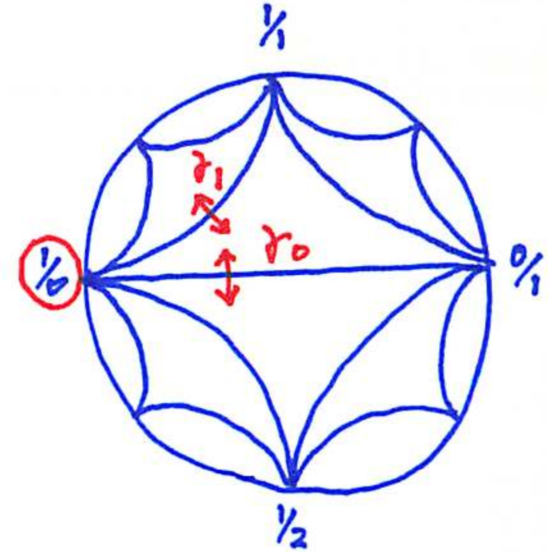
$$\mathcal{M}_0(B^3, t(\infty)) := \{ f \in \mathcal{M}(B^3, t(\infty)) \mid f_* = id \in \mathrm{Out}(\pi_1(B^3 - t(\infty))) \}$$

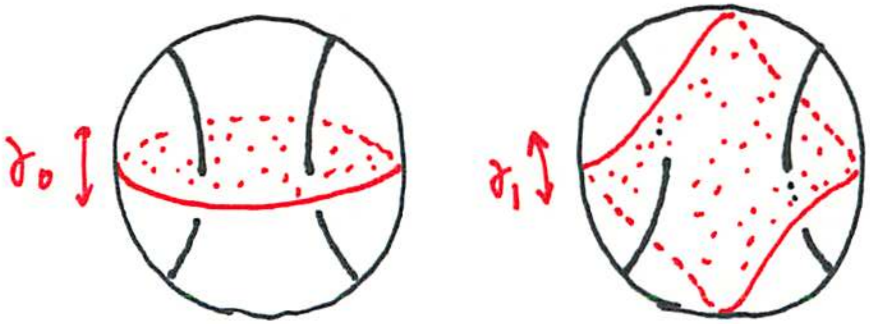
Observation

$$\mathrm{Aut}(D)$$

$$\cup$$

$$\Gamma_\infty := \Phi(\mathcal{M}_0(B^3, t(\infty))) = \left\langle \begin{array}{l} \text{reflections in the edges of } D \\ \text{with endpoint } \infty \end{array} \right\rangle$$



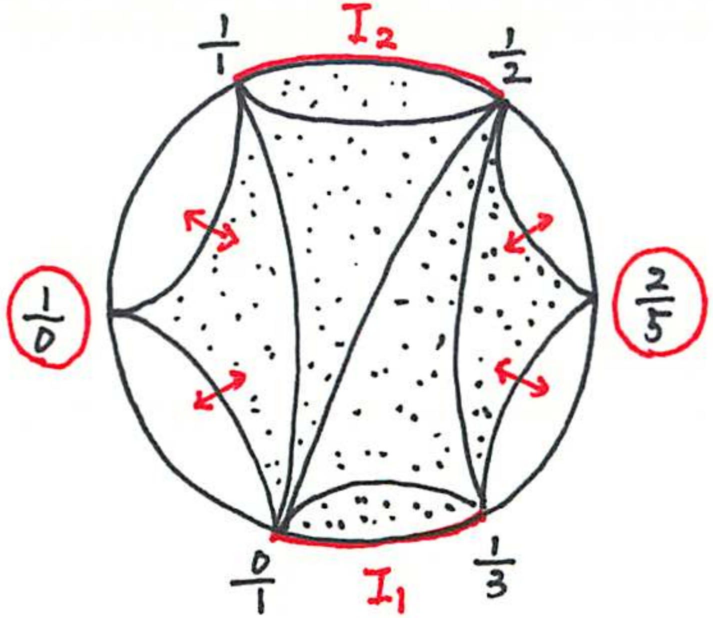


Similarly

$$\begin{array}{c} \mathrm{Aut}(D) \\ \cup \end{array}$$

$$\Gamma_r := \Phi(\mathcal{M}_0(B^3, t(r))) = \left\langle \begin{array}{l} \text{reflections in the edges of } D \\ \text{with an endpoint } r \end{array} \right\rangle$$

Consider $\hat{\Gamma}_r := \langle \Gamma_\infty, \Gamma_r \rangle < \mathrm{Aut}(D)$



- The limit set $\Lambda(\hat{\Gamma}_r)$ = closure of $\hat{\Gamma}_r\{\infty, r\}$
- $I_1 \cup I_2$ is a fundamental domain of the action of $\hat{\Gamma}_r$ on the domain of discontinuity $\Omega(\hat{\Gamma}_r) := \partial \mathbb{H}^2 - \Lambda(\hat{\Gamma}_r)$

Observation [Ohtsuki - Riley - S]

(1) For any $s \in \hat{\mathbb{Q}}$, there is a unique $s_0 \in I_1 \cup I_2 \cup \{\infty, r\}$

st $s = \gamma(s_0)$ for some $\gamma \in \hat{\Gamma}_r$

(2) $\alpha_s \sim \alpha_{s_0}$ in $S^3 - K(r)$

(3) If $s_0 = \infty$ or r, then $\alpha_s \sim 1$ in $S^3 - K(r)$

(Proof of (2))

- If $s = \gamma(s_0)$ with $\gamma \in \Gamma_\infty$,

  then $\alpha_s \sim \alpha_{s_0}$ in $B^3 - t(\infty)$ and so in $S^3 - K(r)$.

- If $s = \gamma(s_0)$ with $\gamma \in \Gamma_r$

  then $\alpha_s \sim \alpha_{s_0}$ in $B^3 - t(r)$ and so in $S^3 - K(r)$.

Question Is the converse true ?

[Lee - S : arXiv : 1004.2571]

$\alpha_s \sim 1$ in $S^3 - K(r)$ iff $s \in \hat{\Gamma}_r \{\infty, r\}$.

i.e, if $s \in I_1 \cup I_2$, then $\alpha_s \not\sim 1$ in $S^3 - K(r)$.

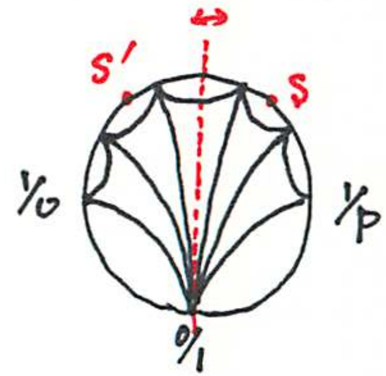
[Lee-S : arXiv : 1010.2232, 1103.0856, 1111.3562]

$\alpha_s \sim \alpha_{s'}$ in $S^3 - K(r)$ for distinct $s, s' \in I_1 \cup I_2$,

iff one of the following holds :

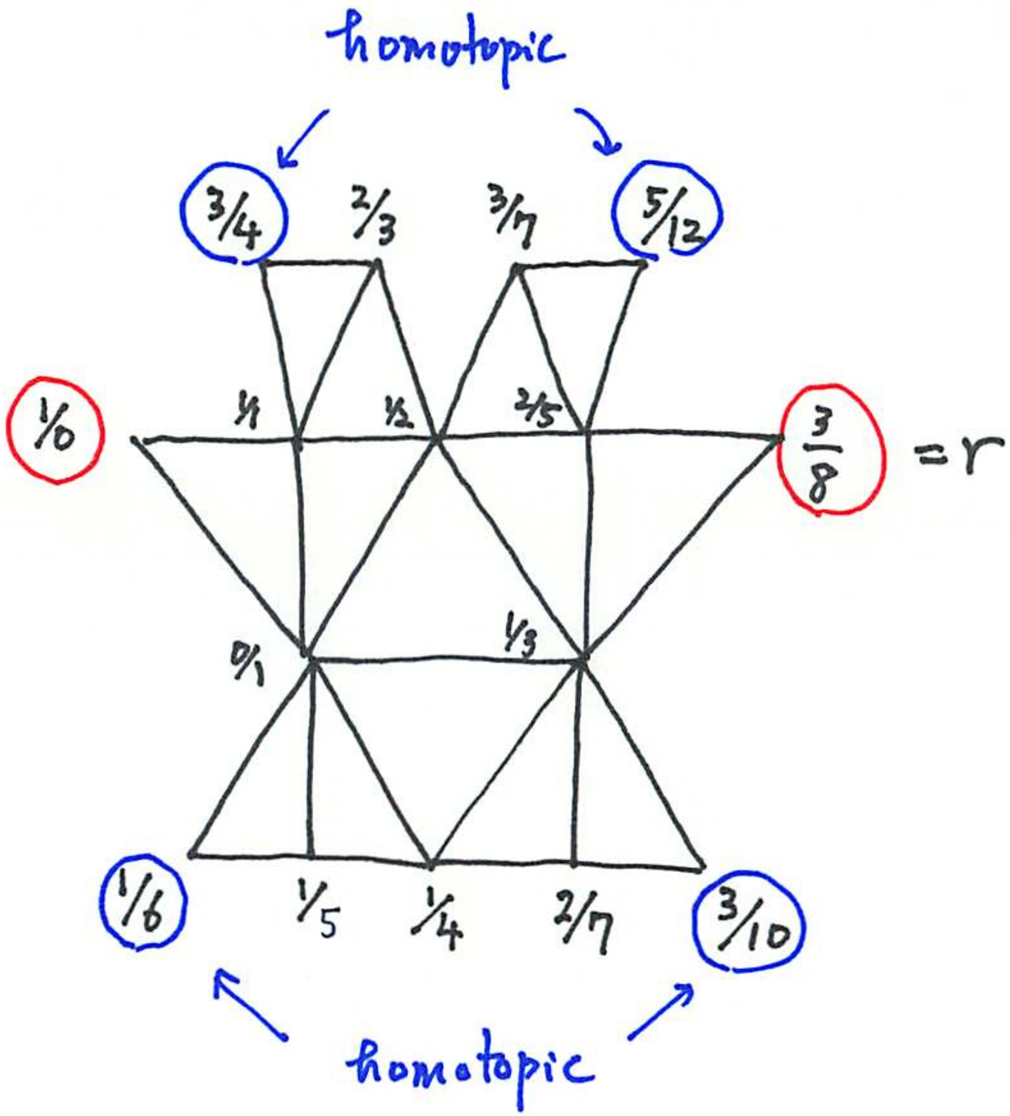
(1) $r = 1/p$ (2-bridge torus link) and $s = q_1/p_1$, $s' = q_2/p_2$

st $q_1 = q_2$ and $q_1/(p_1+p_2) = 1/p$.



(2) $r = 3/8$ (Whitehead link) and

$\{s, s'\} = \{1/6, 3/10\}$ or $\{3/4, 5/12\}$.



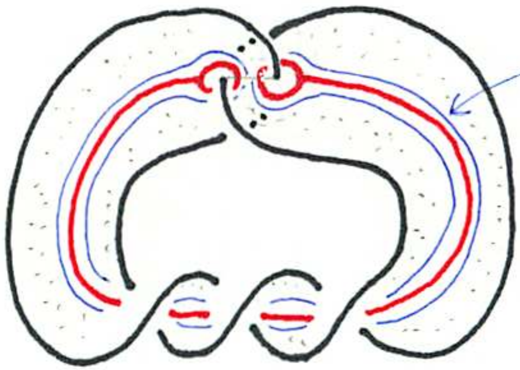
homotopic
3/4
2/3
3/7
5/12
1/0
1/1
1/2
2/5
3/8
= r
0/1
1/3
1/6
1/5
1/4
2/7
3/10
homotopic

[Peripheral Problem]

For a hyperbolic 2-bridge link $K(r)$,
the loop α_s $(s \in I_1 \cup I_2)$ is peripheral,
iff one of the following holds.

(1) $r = 2/5$ and $s = 1/5$ or $3/5$.

(2) $r = n/(2n+1)$ and $s = n+1/2n+1$



This peripheral loop in $S^3 - K(n/2n+1)$ is isotopic to $\alpha_{n+1/2n+1}$ in $S^3 - K(n/2n+1)$.

[Primitiveness Problem]

For a hyperbolic 2-bridge link $K(r)$,
the loop α_s $(s \in I_1 \cup I_2)$ is not primitive
iff one of the following holds.

(1) $r = 2/5$ and $s = 2/7$ or $3/4$.
In this case, $\alpha_s = \beta^3$ for some primitive $\beta \in G(K(r))$.

(2) $r = 3/7$ and $s = 2/7$.
In this case, $\alpha_s = \beta^2$ for some $\beta \in G(K(r))$.

Question

For a hyperbolic link $K(r)$,
when the loop α_s is isotopic to a simple closed geodesic ?

(Idea of Proof)

- Starting point is :

  [Keen - Series], [Komori - Series]

  (0) $\alpha_s \sim 1$ in $B^3 - t(\infty)$ iff $s = \infty$

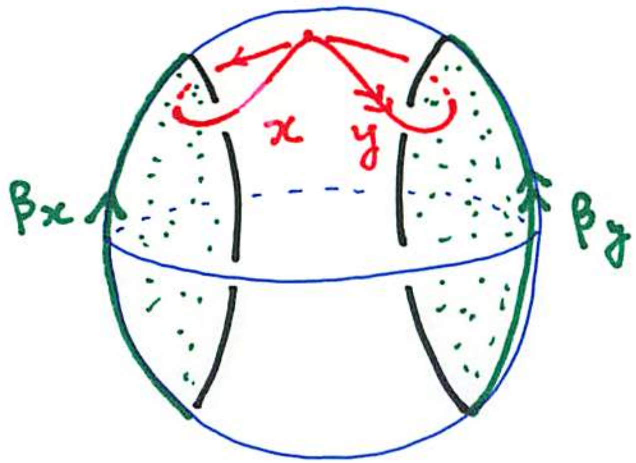
  (1) $\alpha_s \sim \alpha_{s'}$ in $B^3 - t(\infty)$ iff $s' \in \Gamma_\infty \cdot s$

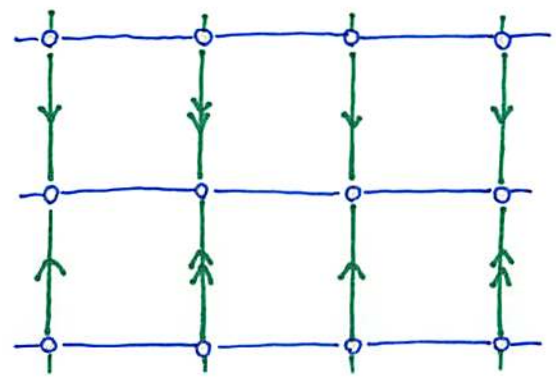
- Key tool is the Small Cancellation Theory, where analysis of the "cutting sequence" of a straight line in $\mathbb{R}^2$ plays a crutial role.

  cf. [Series : The geometry of Markoff numbers]

Upper presentation of $G(K(r)) = \pi_1(S) / \langle\langle \alpha_\infty, \alpha_r \rangle\rangle = \pi_1(B^3 - t(\infty)) / \langle\langle \alpha_r \rangle\rangle$

$\pi_1(B^3 - t(\infty)) = \langle x, y \rangle$

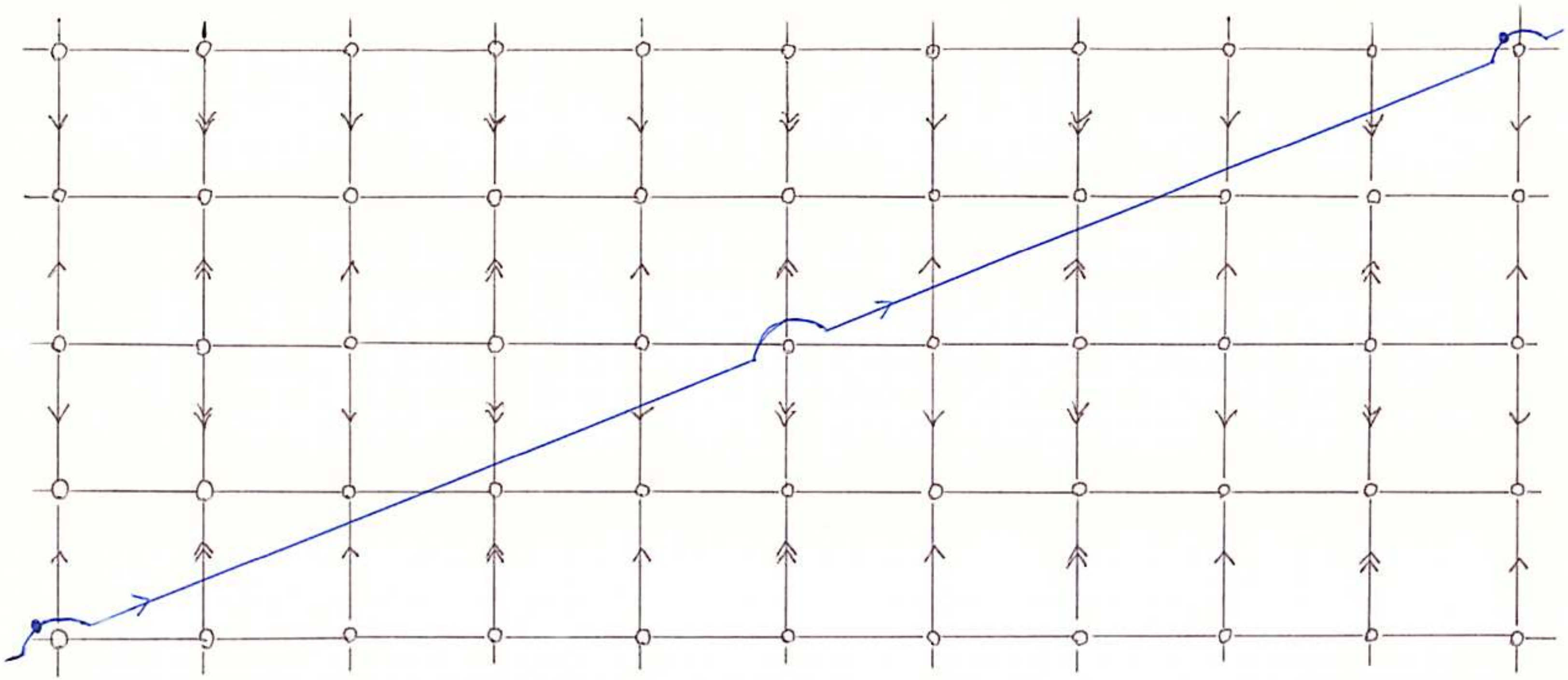




For a loop $\alpha \subset S$,

$[\alpha] \in \pi_1(B^3 - t(\infty)) = \langle x, y \rangle$

is obtained by "reading" the intersections of α with β_x and β_y.



$$[\alpha_{2/5}] := u_{2/5} = x \cdot y x \bar{y} \bar{x} \cdot y \cdot x y \bar{x} \bar{y}$$

$$= x y x \cdot \bar{y} \bar{x} \cdot y x y \cdot \bar{x} \bar{y}$$

$S(2/5) := S(u_{2/5}) := (3, 2, 3, 2)$ S-sequence

$CS(2/5) := ((3, 2, 3, 2))$ Cyclic S-sequence

Observation

- $u_{2/5} = x\, y\, x\, \bar{y}\, \bar{x}\, y\, x\, y\, \bar{x}\, \bar{y}$ is alternating, i.e. x and y apper alternatively.
- $u_{2/5}$ is determined by its S-sequence $(3, 2, 3, 2)$ and the initial letter x.
- Any alternating word w with $S(w) = S(2/5)$ is conjugate to $u_{2/5}$ or $\bar{u}_{2/5}$.

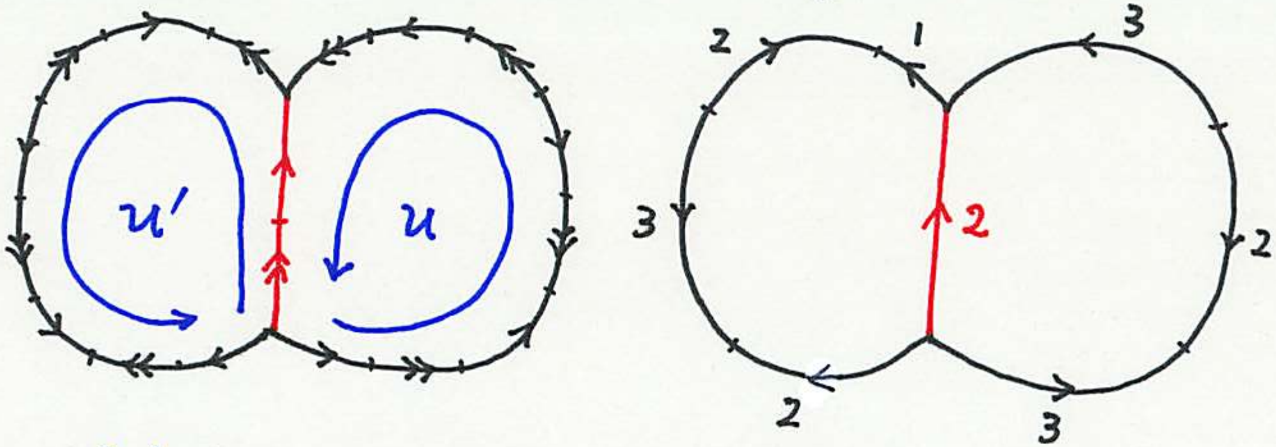
$$x\, y\, x\, \bar{y}\, \bar{x}\, y\, x\, y\, \bar{x}\, \bar{y} = u_{2/5}$$
$$y\, x\, y\, \bar{x}\, \bar{y}\, x\, y\, x\, \bar{y}\, \bar{x} \sim u_{2/5}$$
$$\bar{x}\, \bar{y}\, \bar{x}\, y\, x\, \bar{y}\, \bar{x}\, \bar{y}\, y\, x \sim \bar{u}_{2/5}$$
$$\bar{y}\, \bar{x}\, \bar{y}\, x\, y\, \bar{x}\, \bar{y}\, \bar{x}\, y\, x \sim \bar{u}_{2/5}$$

- Conjugacy class of $\{u_{2/5}, \bar{u}_{2/5}\}$ is determined by $CS(2/5)$.

- $G(K(2/5)) = \langle x, y \mid u_{2/5} \rangle$, $u_{2/5} = x y x \bar{y} \bar{x} y x y \bar{x} \bar{y}$

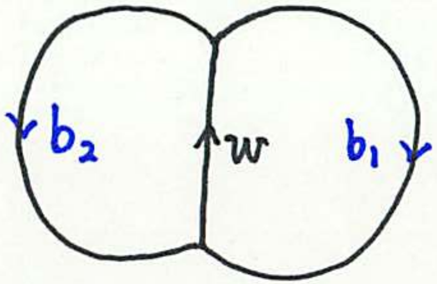
- van Kampen diagram over $\{u_r\}$

  = simply connected 2-dim cell complex in $\mathbb{R}^2$, where each oriented edge is labeled with an element in $F[x, y]$.
  st. the boundary label of a 2-cell is a cyclically reduced word representing the cyclic word $u_r^{\pm 1}$.



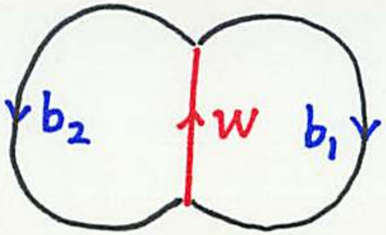
boundary label $= u \cdot u' = x y x \bar{y} \bar{x} y x y \underbrace{\bar{x} \bar{y}}_{\text{piece}} \cdot \underbrace{y x}_{\text{piece}} y \bar{x} \bar{y} x y x \bar{y} \bar{x}$

- **Reducible** pair in a van-Kampen diagram



where $b_1 = b_2$

- A van-Kampen diagram is **reduced** if it has no reduced pairs.



- $b_1 \neq b_2$ though wb_1 and wb_2 are cyclic conjugates of $u_r^{\pm 1}$.
- w is called a **piece**.

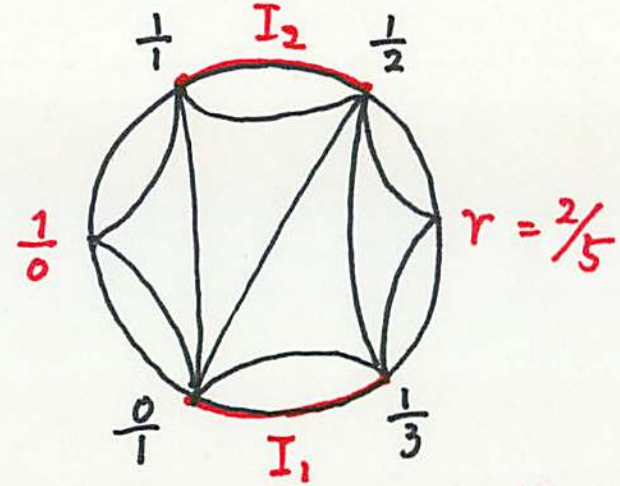
Key Lemma : a complete characterization of pieces for $\{u_r\}$.

Cor $\{u_r\}$ satisfies the condition $C(4)$.
i.e. the cyclic word u_r is not a product of 3 (= 4−1) pieces.

(Proof of "$\alpha_s \neq 1$ in $S^3 - K(r)$ if $s \in I_1 \cup I_2$")

Show that there is no reduced van-Kampen diagram with boundary label α_s ($s \in I_1 \cup I_2$).

Intuition behind the proof

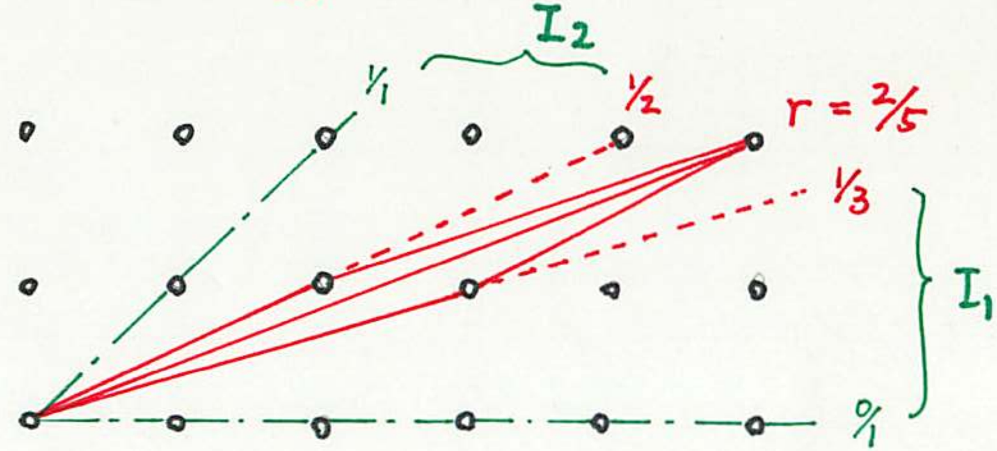


$s \in I_1 \cup I_2$

$\Leftrightarrow$ The slope s is far from ∞ and r

$\Leftrightarrow$ α_s and $u_r = \alpha_r$ cannot share a long subword.

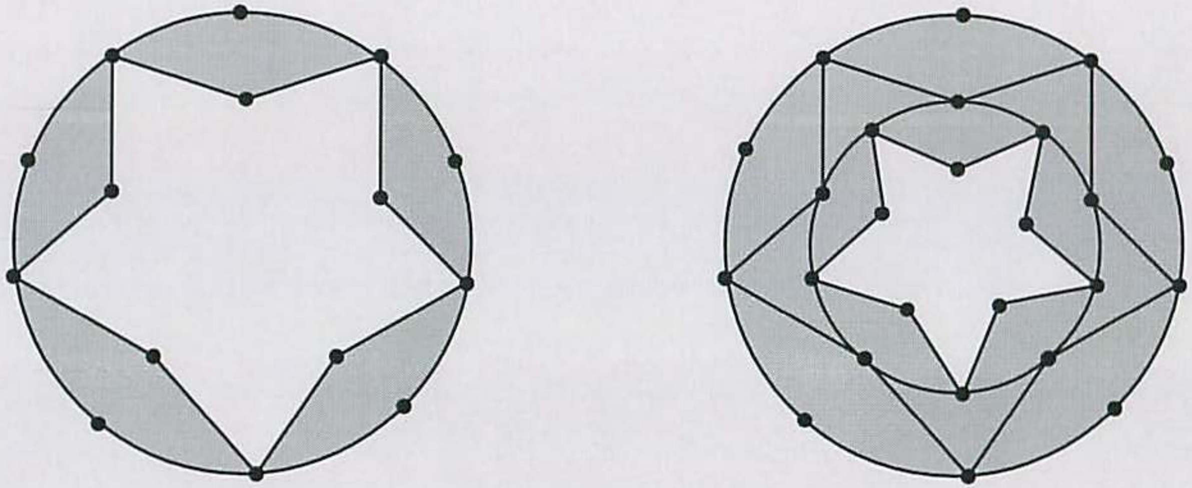
ie α_s admits only small cancellations.



(Proof of Conjugacy Theorem)

Structure Theorem

The following illustrates the only possible annular diagrams between α_s and $\alpha_{s'}$ with $s, s' \in I_1 \cup I_2$.



The proof of Conjugacy Thm is divided into the following 3 cases.

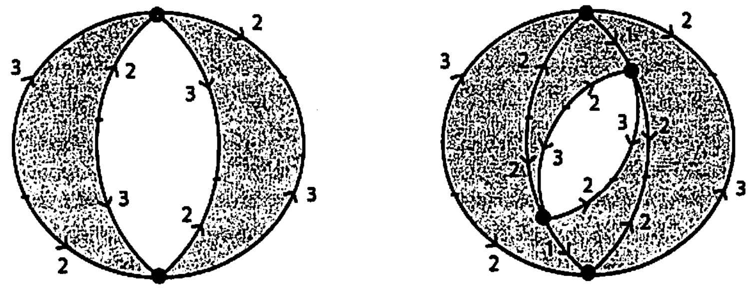
(I) $r = 1/p$

(II) $r = [m, n]$ or $[m, 2, n]$ (various exceptional homotopies)

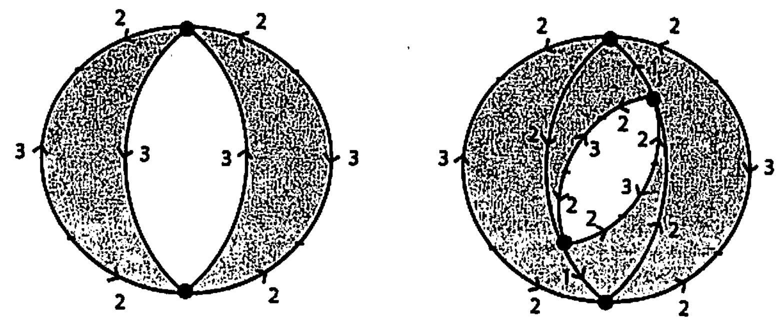
(III) Inductive argument for general cases.

Remark The figure-eight knot case is the most complicated!

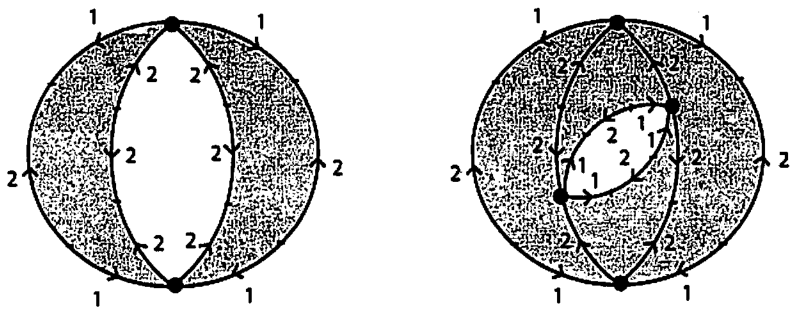
Annular diagrams for $G(K(3/5))$



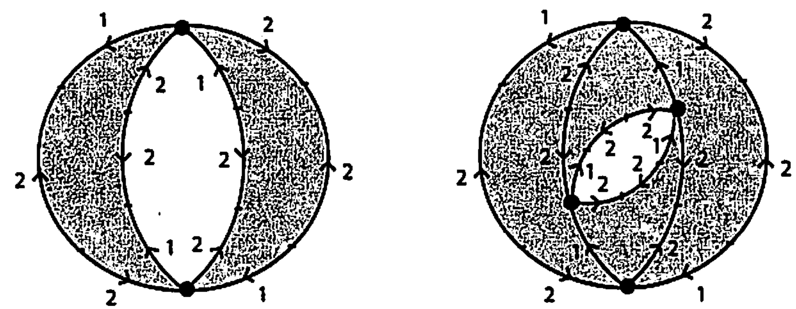
$u_{1/6}$ is commutative with a meridian, and so peripheral



$u_{2/7} \sim (yx)^3$

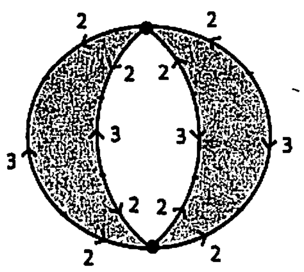


$u_{3/4} \sim (\bar{y}\,\bar{x}\,y\,x)^3$



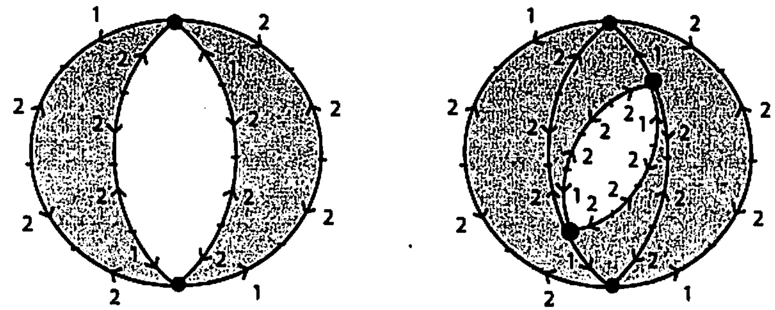
$u_{3/5}$ is peripheral, because it is commutative with a meridian

Annular diagram for $G(K(\frac{n}{2n+1}))$ $(n \geq 3)$



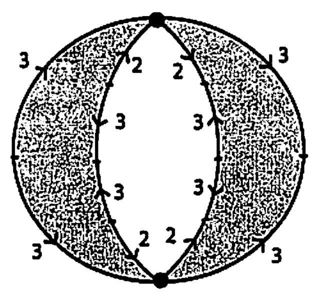
$u_{3/7} = (x y^2 x \bar{y} \bar{x} \bar{y})^2$

in $G(K(3/7))$



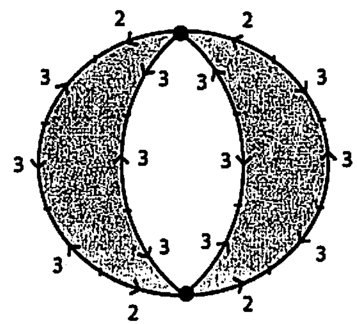
$u_{\frac{n+1}{2n+1}}$ is peripheral, because it is commutative with a meridian.

Annular diagram for $G(K(3/8))$



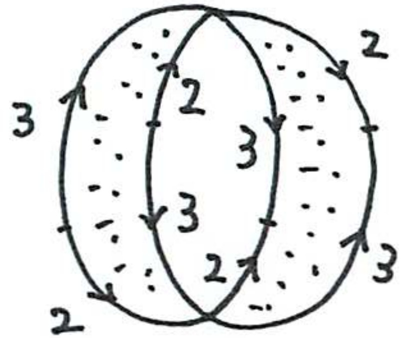
$u_{1/6} \sim u_{3/8}$

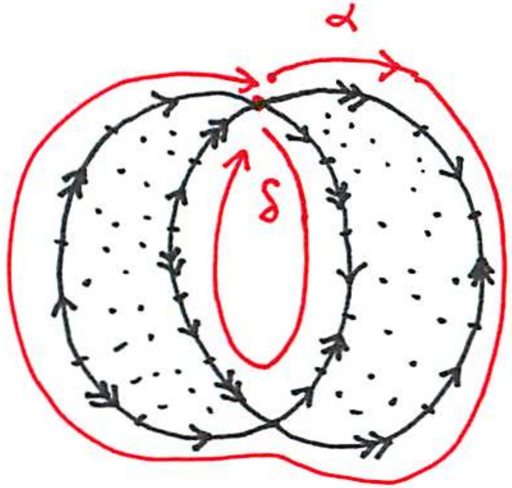
in $G(K(3/10))$



$u_{3/4} \sim u_{5/12}$

in $G(K(3/8))$





Outer boundary label $\alpha = y\,x\,\bar{y}\,\bar{x}\,\bar{y}\,\bar{x}\,\bar{y}\,x\,y\,x$

$= \bar{x}\,\bar{y}\,\bar{x}\,(x\,y\,x\,y\,x\,\bar{y}\,\bar{x}\,\bar{y}\,\bar{x}\,\bar{y})\,x\,y\,x$

$= w_1\; u_{1/5}\; \overline{w_1} \qquad (w_1 = \bar{x}\,\bar{y}\,\bar{x})$

Inner boundary label $\delta = x\,y\,x\,\bar{y}\,\bar{x}\,\bar{y}\,\bar{x}\,\bar{y}\,x\,y$

$= \bar{y}\,\bar{x}\,(x\,y\,x\,y\,x\,\bar{y}\,\bar{x}\,\bar{y}\,\bar{x}\,\bar{y})\,x\,y$

$= w_2\; u_{1/5}\; \overline{w_2} \qquad (w_2 = \bar{y}\,\bar{x})$

Hence $w_1\, u_{1/5}\, \overline{w_1} = w_2\, u_{1/5}\, \overline{w_2}$

So $u_{1/5}$ is commutative with $\overline{w_1}\, w_2 = (x\,y)\,x\,(\bar{y}\,\bar{x})$ = a meridian

Hence $u_{1/5}$ is peripheral.

Speculation following [Minsky, Geom. Top. Monograph 12]

- $(S^3, K) = (B_1^3, t_1) \cup_S (B_2^3, t_2)$ n-bridge decomposition
- $\mathcal{M}(S) = \pi_0 \mathrm{Diff}(S)$ where $S = \partial B_i^3 - t_i$

  $\cup$

  $\mathcal{M}_0(B_i^3, t_i) := \{ f \in \pi_0 \mathrm{Diff}(B_i^3, t_i) \mid f_* = id \in \mathrm{Out}(\pi_1(B_i^3 - t_i)) \}$

  $\Gamma := \langle \mathcal{M}_0(B_1^3, t_1), \mathcal{M}_0(B_2^3, t_2) \rangle \subset \mathcal{M}(S)$
- $\mathcal{C}^{(0)}(S) = \{\text{essential simple loops on } S\}/\text{isotopy}$

  $\cup$

  $\Delta_i := \{\text{the boundaries of essential disks in } B_i^3 - t_i\}$

  $\Delta := \Delta_1 \cup \Delta_2$

Observation If $\alpha \in \Gamma \cdot \Delta$, then $\alpha \sim 1$ in $S^3 - K$

Question Is the converse true ?

[Masur] $\mathcal{M}_0(B_i^3, t_i) \curvearrowright \mathcal{PML}(S)$ has a non-empty domain of discontinuity.

Question Suppose the bridge decomposition is "sufficiently complicated".

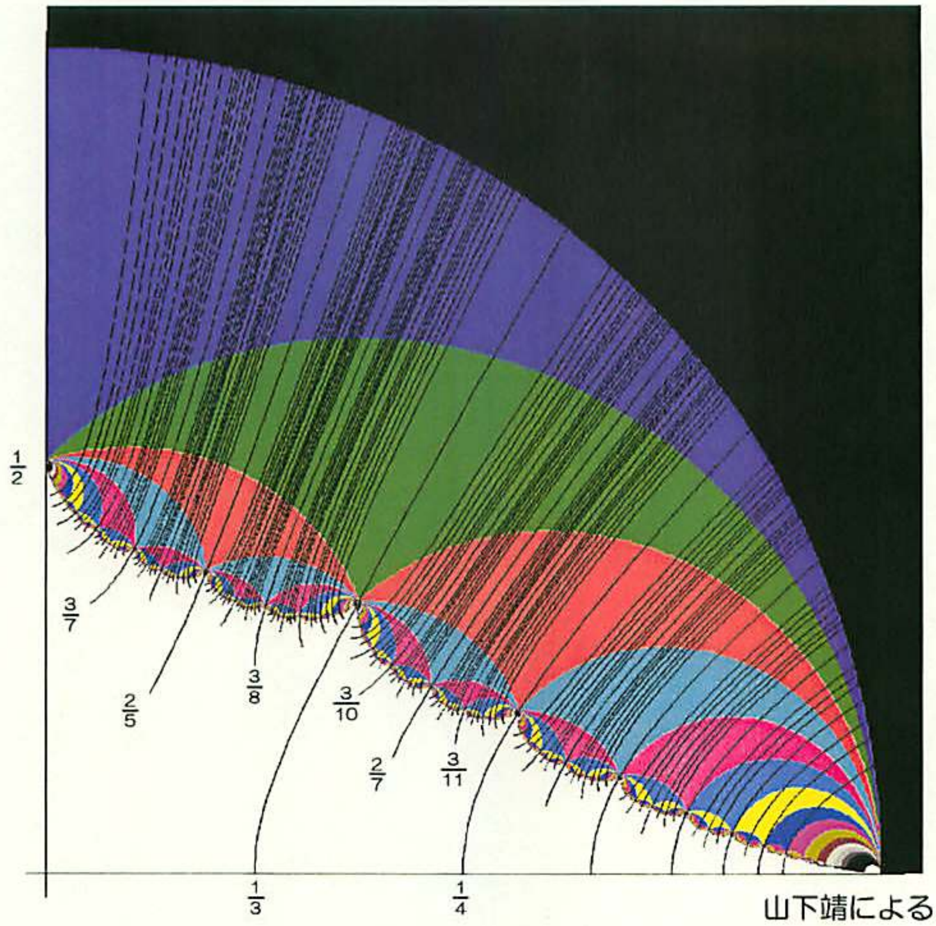
(1) Does $\Gamma = \langle \mathcal{M}_0(B_1^3, t_1), \mathcal{M}_0(B_2^3, t_2) \rangle \curvearrowright \mathcal{PML}(S)$ have a non-empty domain of discontinuity?

(2) $\Gamma \cong \mathcal{M}_0(B_1^3, t_1) * \mathcal{M}_0(B_2^3, t_2)$?

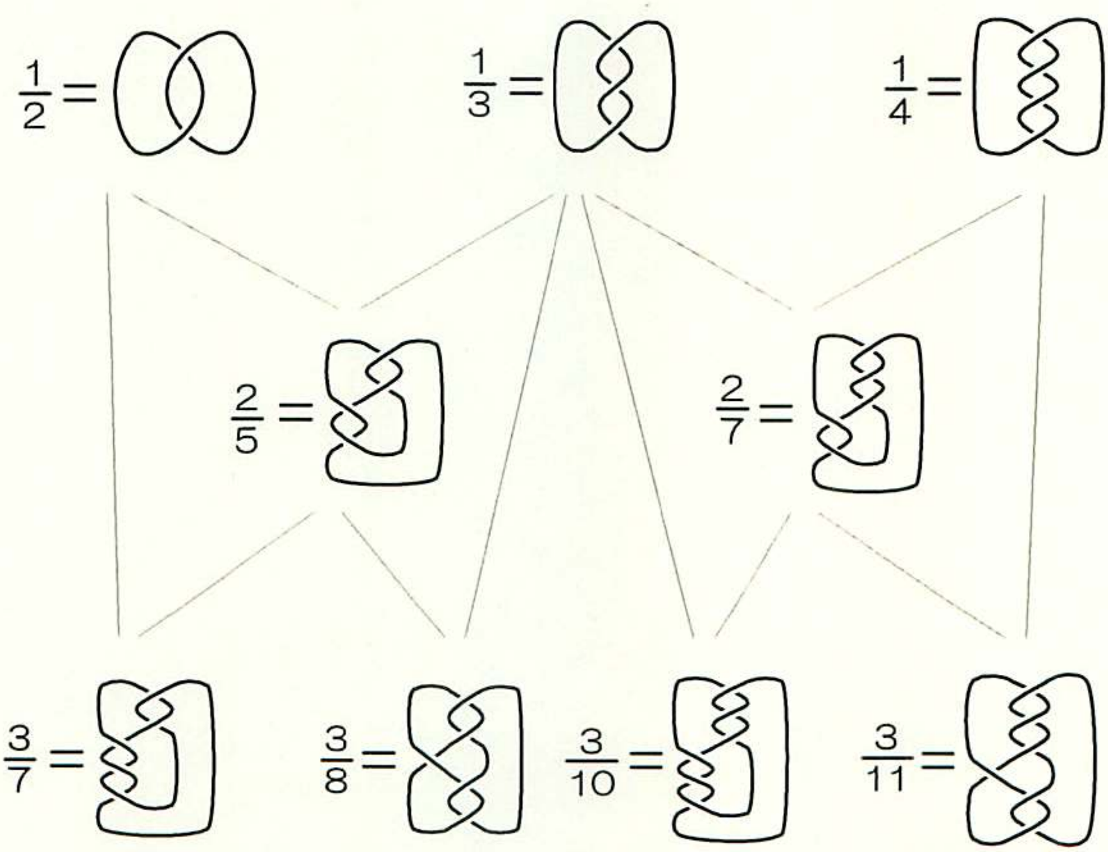
(3) Suppose $\alpha \in \mathcal{C}^{(0)}(S)$ is contained in the domain of discontinuity $\Omega(\Gamma) \subset \mathcal{PML}(S)$. Then can $\alpha \sim 1$ in $S^3 - K$?

(4) Does $cl\{\alpha \in \mathcal{C}^{(0)}(S) \mid \alpha \sim 1 \text{ in } S^3 - K(r)\} \subset \mathcal{PML}(S)$ have measure 0 ?

ライリー切片



山下靖による



Conjectural characterization of the 2-bridge Kleinian groups

$$\mathcal{R} := \{\rho : \pi_1(T) \to PSL(2.\mathbb{C}) \mid \text{type-preserving}\} / \text{conj}$$

$\cup$

$$\mathcal{D} := \{\text{discrete faithful}\} = \overline{\mathcal{QF}} \cong \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \text{diag}(\partial \mathbb{H}^2)$$

↑ Minsky's ending lamination theorem

$\cup$

$$\mathcal{QF} := \{\text{quasi-fuchsian}\} \cong \mathbb{H}^2 \times \mathbb{H}^2 .$$

$\cup$ $\qquad\qquad\qquad\qquad \cup$

$$\mathcal{F} := \{\text{fuchsian}\} \cong \text{diagonal}(\mathbb{H}^2 \times \mathbb{H}^2) \cong \mathbb{H}^2$$

$$\mathcal{D}_{finite} := \{\text{discrete (non-faithful) co-finite volume}\}$$

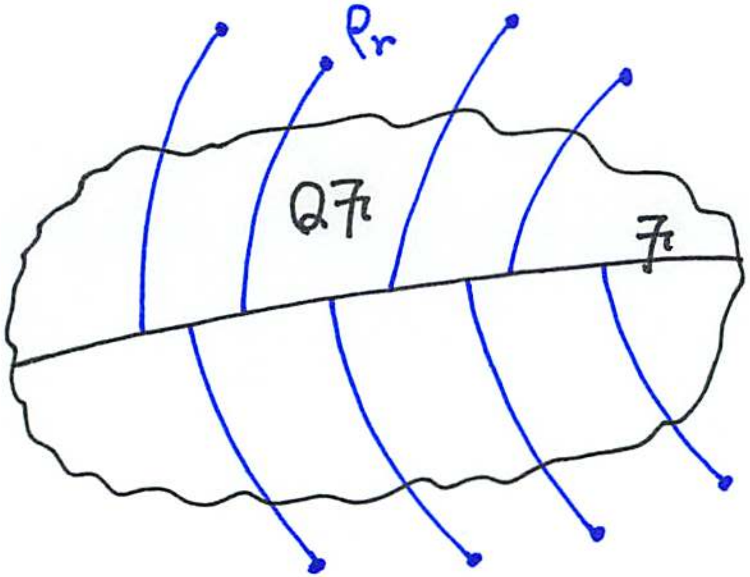
$\cup$? (identical ?)

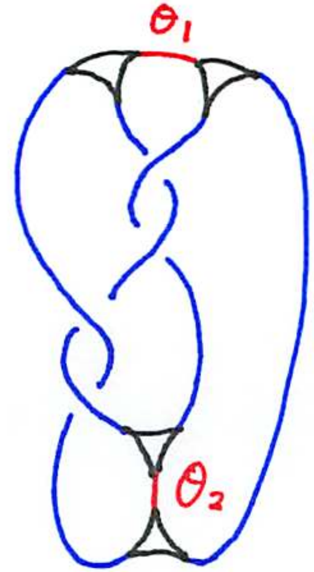
$$\{\rho_r \mid r = q/p \quad q \not\equiv \pm 1 \ (p)\} = \{\text{2-bridge Kleinian group}\}$$

[Akiyoshi - S - Wada - Yamashita]

There is a natural "path" $\{\rho_{r,\theta_1,\theta_2} \mid 0 \le \theta_i \le 2\pi\}$

from $Q\mathcal{F}$ to ρ_r





[Conjecture]

$\rho \in R$ is discrete iff $\rho \in \partial\mathcal{D} = \overline{Q\mathcal{F}}$

or $\rho = \rho_{r,\, 2\pi/m,\, 2\pi/n}$