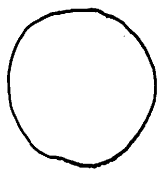
Knot group.

- the knot group of the trivial knot



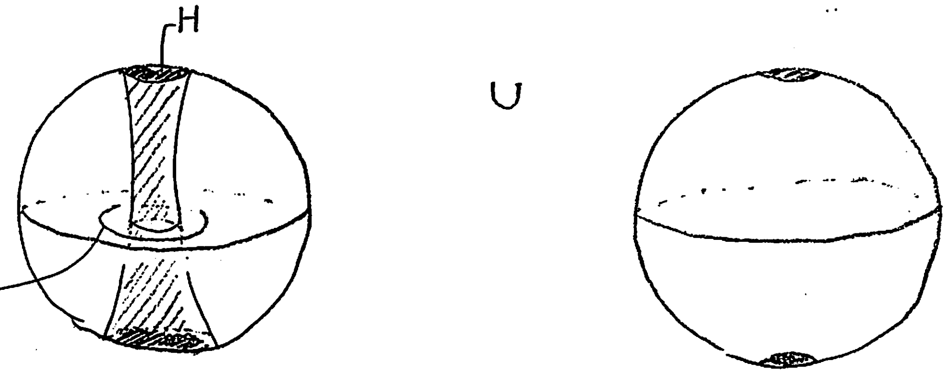
O

$$G(O) = \pi_1(S^3 - \mathring{N}(O)) \cong \mathbb{Z}.$$

proof.

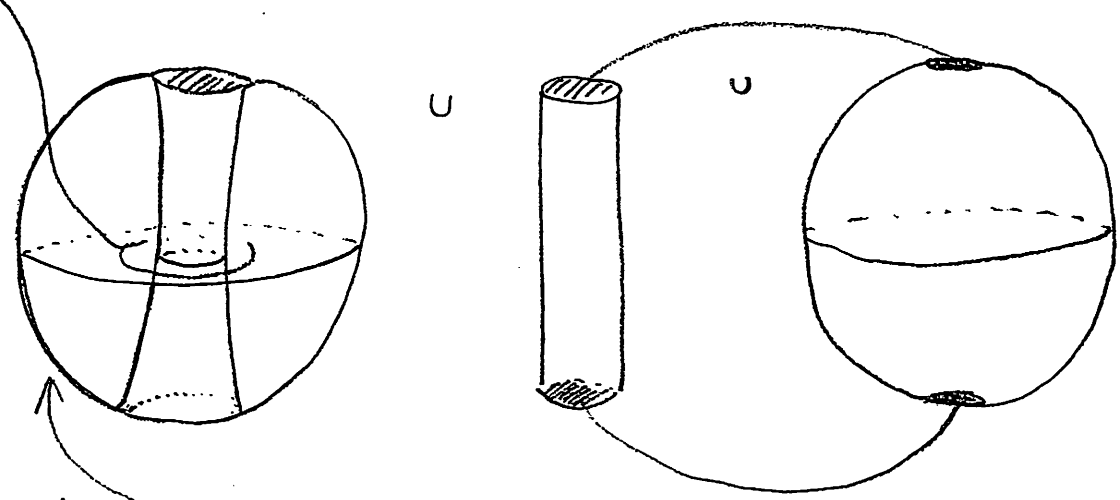
$$S^3 = B_+^3 \cup B_-^3$$

$S^3 =$



the trivial knot

$=$



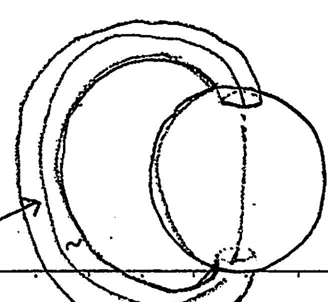
attach $\cong$

$$= cl(B_+^3 - H) \cup (H \cup B_-^3)$$

$\shortparallel$

$N(O)$.

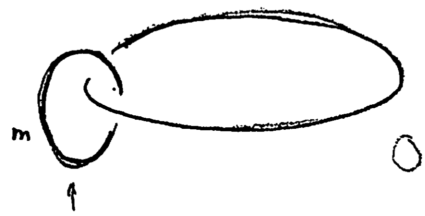
both solid tori



a gen. of $G(O)$

$E(0) = H \cup B^3_- \cong S^1 \times D^2$.

In $\mathbb{R}^3$ picture :



a generator of $G(0)$

: meridian.

<u>Remark</u> m bounds a disk D so that $D \cap O$ = a pt.

Theorem (A consequence of Dehn's Lemma, Loop theorem).

K : trivial $\Longleftrightarrow$ $G(K) \cong \mathbb{Z}$.

In the above proof, we showed

$$S^3 = \underbrace{N(0) \cup E(0)}_{\text{both solid tori } D^2 \times S^1}$$

This is viewed as follows : (Hopf fibration)

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1 \}$$
$$\cup$$
$$T_r := \{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 = 1 - r^2,\ |z_2|^2 = r^2 \} \qquad 0 \le r \le 1.$$

We obtain :

$$S^3 = \coprod_{0 \le r \le 1} T_r$$

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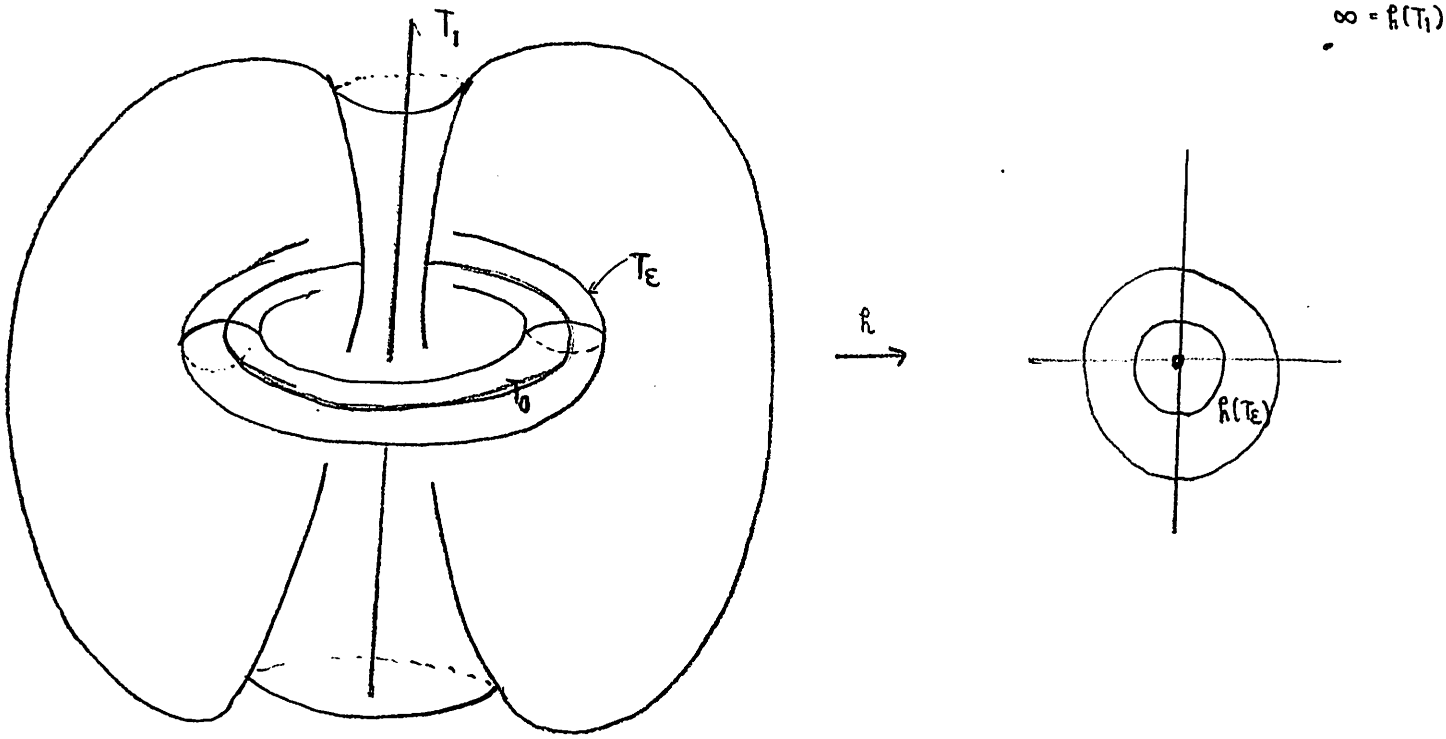
$S^3 = \mathbb{R}^3 \cup \{\infty\}$. $\infty \leftrightarrow (0,1) \in S^3 = \{(z_1,z_2)\in\mathbb{C}^2 \mid |z_1|^2+|z_2|^2=1\}$.

$T_r = S^1 \times S^1 \qquad 0<r<1$

$S_1^1 := \{(z_1,0) \mid |z_1| = 1\} \qquad r=0$

$S_2^1 := \{(0,z_2) \mid |z_2| = 0\} \qquad r=1$



Hopf fibration

$$h: \underset{\Psi}{S^3} \to \underset{\Psi}{\mathbb{C}\cup\{\infty\}} = \mathbb{CP}^1 \qquad (\text{Recall } \infty \leftrightarrow (0,1)).$$

$$(z_1,z_2) \mapsto \frac{z_2}{z_1}$$

$$S^3 = \coprod_{w\in\mathbb{C}\cup\{\infty\}} h^{-1}(w) \qquad h^{-1}(w) = \{(z_1,z_2)\in\mathbb{C}^2 \mid z_2 = w z_1,\ |z_1|^2 = \frac{1}{1+|w|^2}\} \approx S^1$$

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<u>Exe</u> Depictate $S^3 = \coprod_{w \in \mathbb{C} \cup \{\infty\}} h^{-1}(w)$.

Remark

$h(T_r) = \{ w \in \mathbb{C} \mid |w| = \sqrt{\frac{r^2}{1-r^2}} \}$.

$(\because)$ $(z_1, z_2) \in T_r \iff |z_1| = \sqrt{1-r^2}, \quad |z_2| = r$

$\iff \frac{|z_2|}{|z_1|} = \sqrt{\frac{r^2}{1-r^2}}$

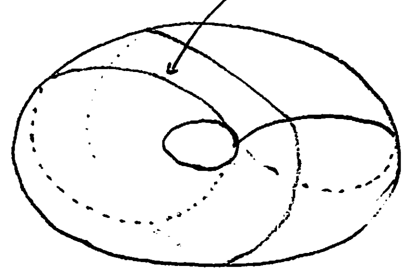
$h| : T_r \to \{ w \in \mathbb{C} \mid |w| = \sqrt{\frac{r^2}{1-r^2}} \}$

Examine $h|T_r$: via S^1-action given by :

$\omega \in S^1 \subset \mathbb{C}$ $\qquad S^3 \to S^3$ $\qquad$ rotation in the directions S^1_1, S^1_2 by the angle θ.

$\omega = e^{2\pi i\theta}$ $\qquad (z_1, z_2) \mapsto (\omega z_1, \omega z_2)$

Look at the fiber of h :



$h(z_1, z_2) = h(z_1', z_2')$

$\iff \frac{z_2}{z_1} = \frac{z_2'}{z_1'}$

$\iff \frac{z_1'}{z_1} = \frac{z_2'}{z_2} \left(:= \omega \in S^1 \right)$

$\iff (z_1', z_2') = \omega (z_1, z_2)$ for some $\omega \in S^1$

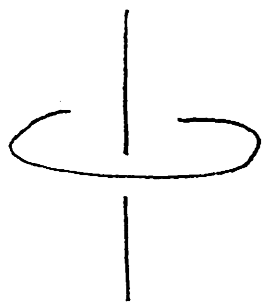
$\iff (z_1, z_2)$ & (z_1', z_2') are on the same orbit.

$$\begin{tikzcd} S^3 \arrow[r, "h"] \arrow[dr, "p"'] & \mathbb{C} \cup \{\infty\} \arrow[d, phantom, "\wr\!\shortparallel"] \\ \arrow[ur, phantom, "\circlearrowleft"] & S^3/S^1 \end{tikzcd}$$

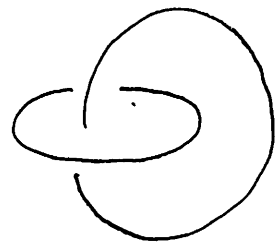
$w_1 \neq w_2 \in \mathbb{C} \cup \{\infty\}$

$\left(S^3, h^{-1}(w_1) \cup h^{-1}(w_2)\right)$

$\cong \left(S^3, h^{-1}(0) \cup h^{-1}(\infty)\right)$



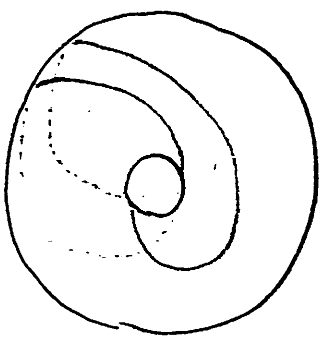
$\cong$



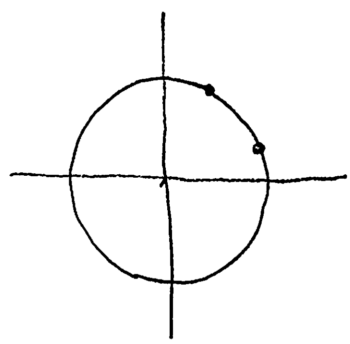
Hopf link

<u>Exe</u>. $W \subset \mathbb{C} \cup \{\infty\}$, $|W| = 3, 4, 5$ Draw $h^{-1}(W)$

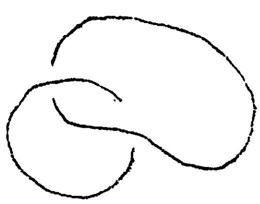
$|W_n| = n$ $W_n \subset \mathbb{C} \cup \{\infty\}$ Draw $h^{-1}(W_n)$



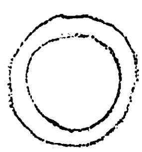
$\xrightarrow{h}$



$\wr\wr$



$\neq$



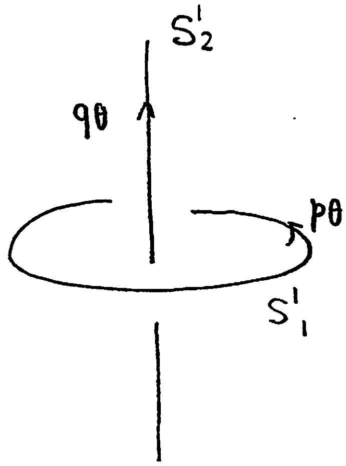
S^1 action generalized and torus knots.

(p,q) : a pair of integers ; mutually prime

$\omega \in S^1$ acts on S^3 by :
$$S^3 \longrightarrow S^3$$
$$(z_1, z_2) \longmapsto (\omega^p z_1, \ \omega^q z_2).$$

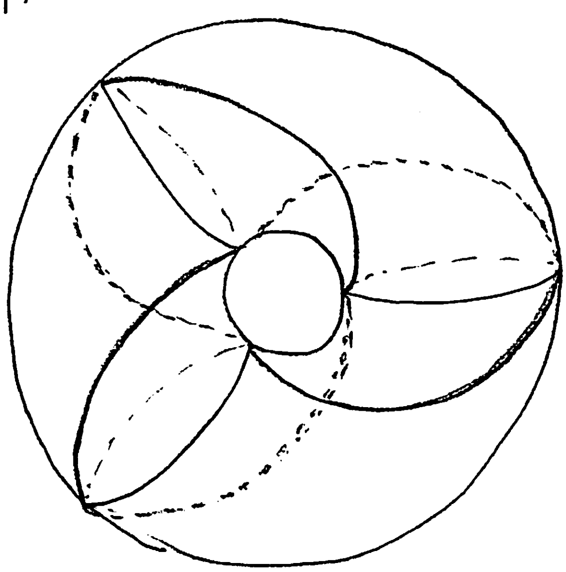
g.c.d. $(p,q) = 1$ implies that the action above is effective
i.e. $\omega \neq 1 \Rightarrow \omega \cdot \neq id_{S^3} : S^3 \circlearrowleft$

$\omega = e^{2\pi i \theta}$



On S^1_1, a rotation by $p\theta$
S^1_2, 〃 $q\theta$.

Example $(p,q) = (2,3)$



$|p| > 1 < |q|$.

- S^1_1, S^1_2 singular orbits
- $K_{p,q}$: regular orbit
  ↑
  torus knot of type (p,q)

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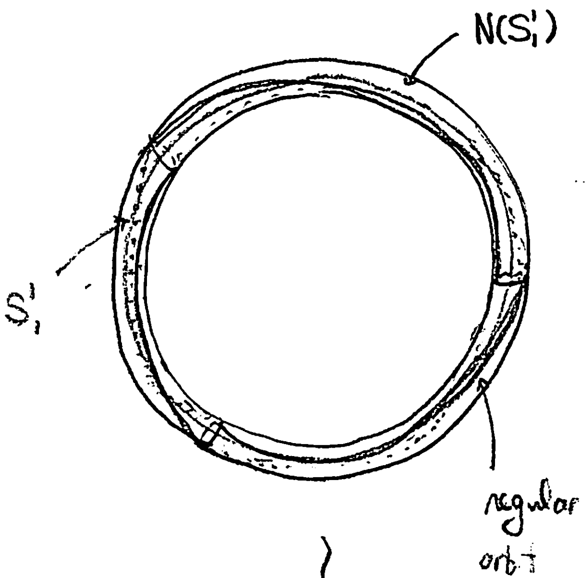
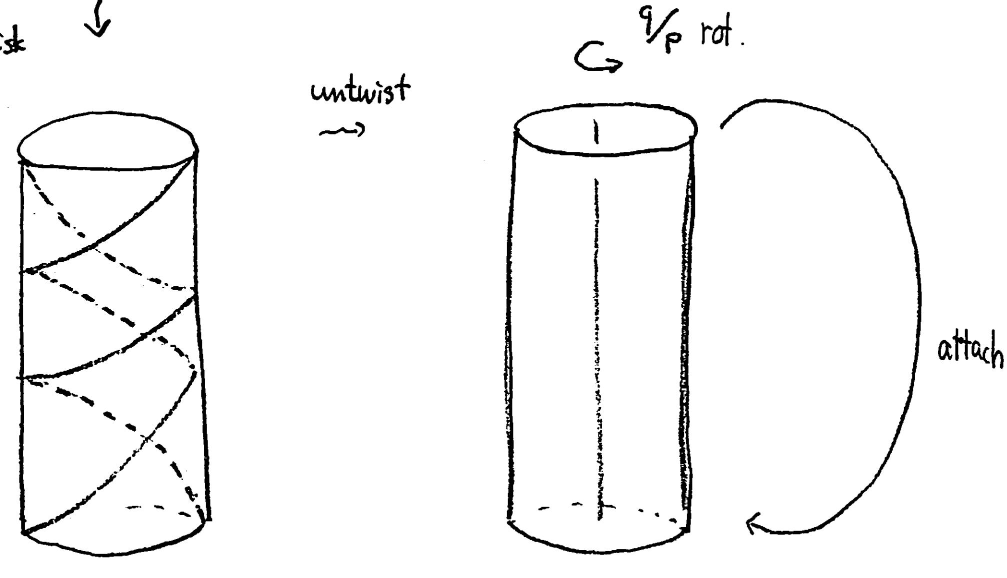
The "fibration" formed by these S^1-orbits is called a (special) <u>Seifert fibration</u>

A local picture of the above fibration on a nbd of S_1^1



Cut by the meridian disk



$N(S^1_1)$

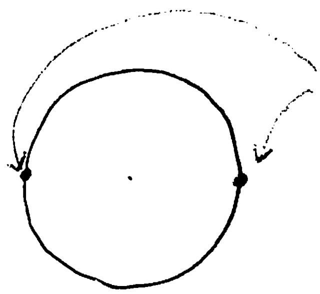
fibration $\{ w \times I \}_{w \in D^2}$ on $D^2 \times I$.

$D^2 \times 0$ and $D^2 \times 1$ are identified after the ration q/p

$\rightsquigarrow$

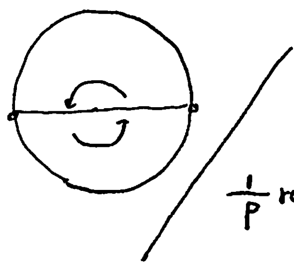
a fibration of $N(S^1_1)$

$N(S^1_1)/S^1$ =



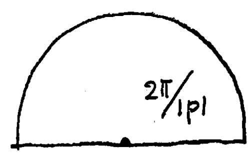
identified since they are in the same orbit. (pic. for $p = 2$).

=



$\frac{1}{p}$ rotation

$\cong$



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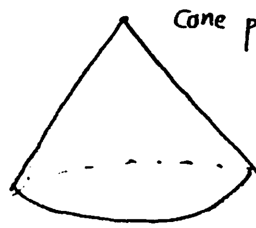
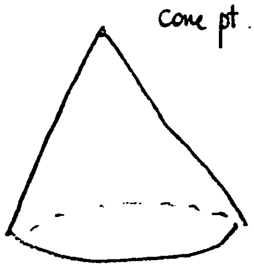


Cone point of cone angle $\frac{2\pi}{|p|}$

$N(S^1_2)/S^1$ =



cone pt. of cone angle $\frac{2\pi}{q}$

No. View $S^3 = N(S^1_1) \cup N(S^1_2)$ and $N(S^1_1)/S^1 \cong$

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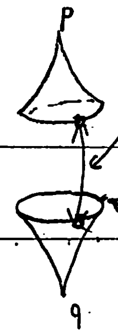
$N(S^1_2)/S^1 \cong$

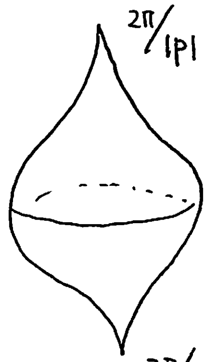


identified

the fiber of this point is a torus knot.

$S^3/S^1 =$

$2\pi/|p|$



$2\pi/|q|$

a 2 dim'l orbifold.

(V-manifold in the sense of Satake)

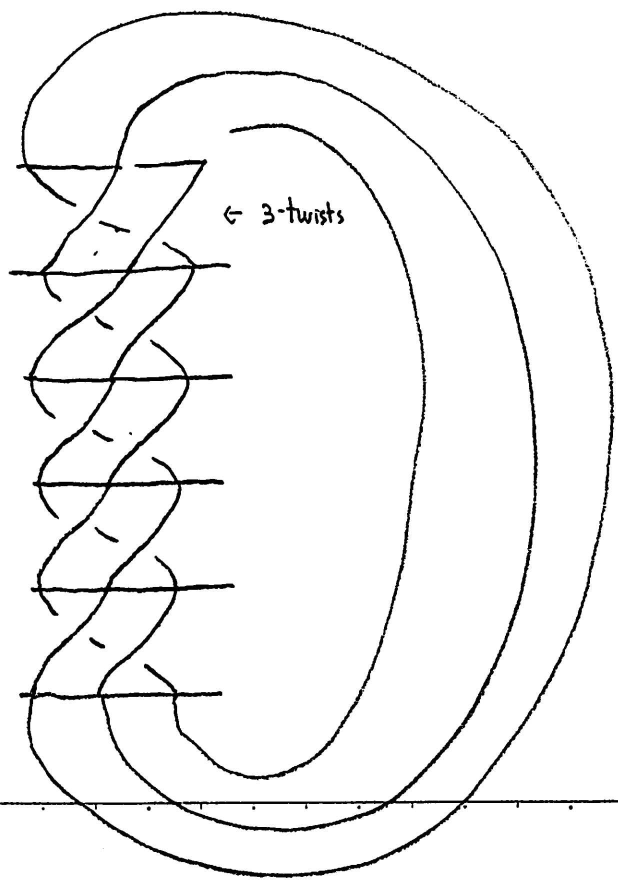
Reference : P. Scott The geometry of 3-manifold Bull. London Math. Soc.

HP.

Def torus knot $K_{p,q}$ = a regular fiber of (p,q) Seifert fibration $h_{p,q}(z_1,z_2) = z_2^{\,q}/z_1^{\,p}$

$p=3$
$q=5$.

5-copies



Picture by closing a braid

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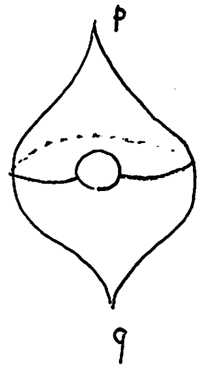
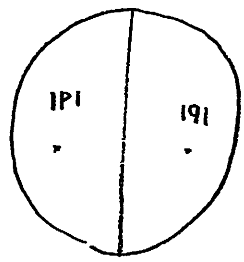
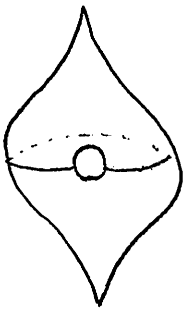
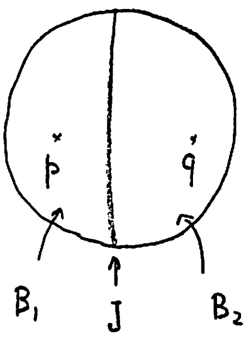
$E(K_{p,q})$: a Seifert fibered space over the 2 dim'l orbifold

$E(K_{p,q}) \xrightarrow{h_{p,q}}$ 

$\cong$ 

Prop $G(K_{p,q}) = \langle a, b \mid a^p = b^q \rangle$

proof

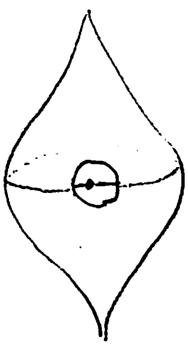
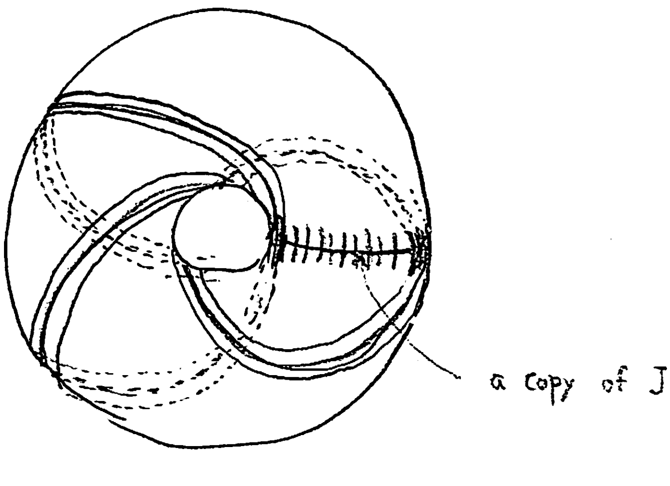

$=$ 

$= B_1 \underset{J}{\cup} B_2$

$E(K_{p,q}) = X_1 \underset{Y}{\cup} X_2$

$X_i = h_{p,q}^{-1}(B_i)$

$Y = h_{pq}^{-1}(J)$


$\leftarrow$ 

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(a regular nbd of $K_{p,q}$ in T^2.)

$$Y = h_{p,q}^{-1}(J) = T^2 \setminus \mathring{N}(K_{p,q} : T^2) \cong S^1 \times J \quad \text{: the annulus}$$

$$\therefore \pi_1(Y) \cong \mathbb{Z} = \langle c \rangle \qquad c : \text{a } (p,q)\text{-torus knot}$$

S^3 is the union :

$$S^3 = N(S^1_1) \underset{\substack{T \\ \cup \\ K_{p,q}}}{\cup} N(S^1_2)$$

$$\begin{aligned} X_i &= h_{p,q}^{-1}(B_i) \\ &= N(S^1_i) \setminus \mathring{N}(K_{p,q}) \\ &= \text{the solid torus } N(S^1_i) \setminus (\text{a tube along } K_{p,q}) \cong N(S^1_i) \end{aligned}$$

Hence. $\pi_1(X_i) \cong \pi_1(N(S^1_i)) = \begin{cases} \langle a \rangle & i=1 \\ \langle b \rangle & i=2. \end{cases}$

The inclusion

$$\begin{array}{ccl} \pi_1(Y) & \longrightarrow & \pi_1(X_i) \\ \cup\!\!\!| & & \cup\!\!\!| \\ c & \longmapsto & \begin{cases} a^p & i=1 \\ b^q & i=2. \end{cases} \end{array}$$

By van Kampen theorem,

$$\pi_1(E(K_{p,q})) = \pi_1(X_1) \underset{\pi_1(Y)}{*} \pi_1(X_2) = \langle a, b \mid a^p = c = b^q \rangle \quad //$$

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Characterization of torus knot groups:

$G := G(K_{p,q}) = \langle a, b \mid a^p = b^q \rangle$. We assume $p, q \geqq 2$

$f := a^p = b^q \in Z(G)$.

$(\because)$ $af = fa$, $bf = fb$. //

In particular, $\underset{\mathbb{Z}}{\overset{\langle f \rangle}{\cong}} \triangleleft Z(G)$.

$$G/\langle f \rangle = \langle a \mid a^p = 1 \rangle * \langle b \mid b^q = 1 \rangle$$
$$\cong (\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/q\mathbb{Z}).$$

Thus

$Z(G/\langle f \rangle) = 1$ i.e $\langle f \rangle = Z(G)$.

Summing up:

$$Z(G(K_{p,q})) = \langle f \rangle \cong \mathbb{Z}.$$

$$1 \to \underset{\mathbb{Z}}{\overset{Z(G(K_{p,q}))}{\cong}} \to G(K_{p,q}) \longrightarrow (\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/q\mathbb{Z}) \to 1.$$

Theorem.

$Z(G(K)) \neq \{1\} \iff K \cong K_{p,q}$ for some p, q.

In other words, if K is not a torus knot, then $G(K)$ is centerless.
(and is far from being abelian)

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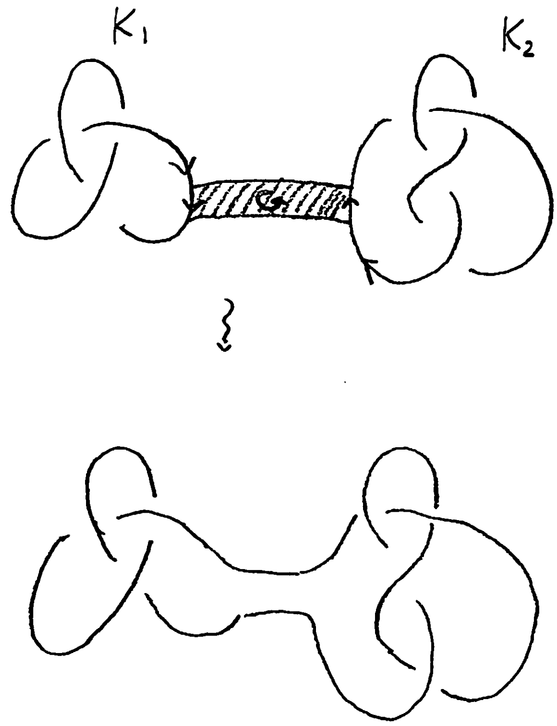
Unique prime factorization of a knot.

Composite knot :



$K_1 \# K_2$

<u>Exe</u>. (1) Find propositions to verify that $K_1 \# K_2$ is well-defined.

(2) Give a definition of the connected sum $(S^3, K_1) \# (S^3, K_2)$

(3) Show that $K_1 \# K_2 \cong K_2 \# K_1$

(4) 〃 $(K_1 \# K_2) \# K_3 \cong K_1 \# (K_2 \# K_3)$.

Def. A knot K is prime $\equiv$ $K = K_1 \# K_2 \Rightarrow (K_1, K_2) = (K, O)$ or (O, K)

(We assume $K \neq O$).

Theorem (H. Schubert)

$\forall K$ knot ($\neq O$)

$$K \cong K_1 \# K_2 \# \cdots \# K_m \qquad K_i : \text{prime}$$

prime factors $K_1, \cdots, K_m$ are uniquely determined (up to enumerations $K_1, \cdots, K_m$)

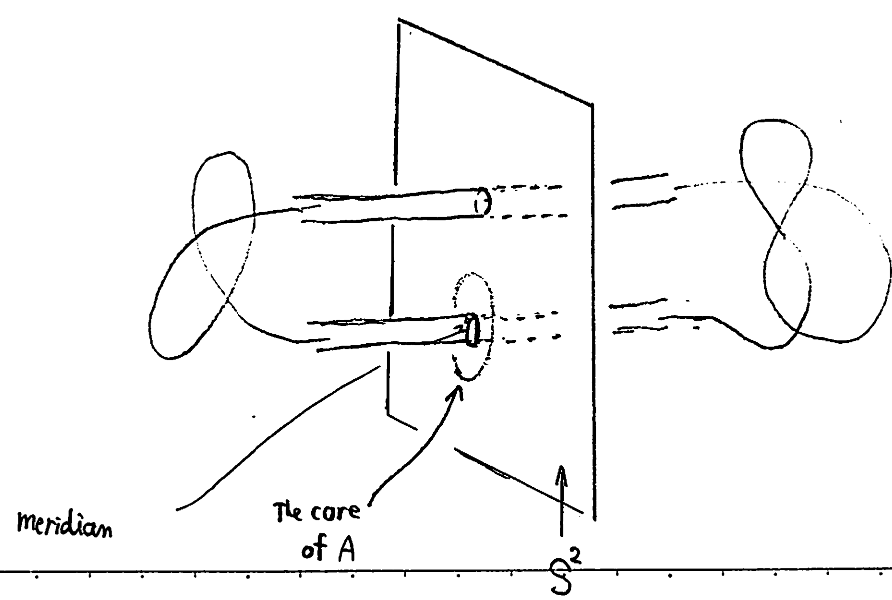
Our goal is to show

$$G(K_1 \# K_2) \cong G(K_1) \underset{\langle m \rangle}{*} G(K_2)$$

m : meridian

proof

$$E(K_1 \# K_2) = E(K_1) \underset{A}{\cup} E(K_2) \qquad A : \text{annulus}$$



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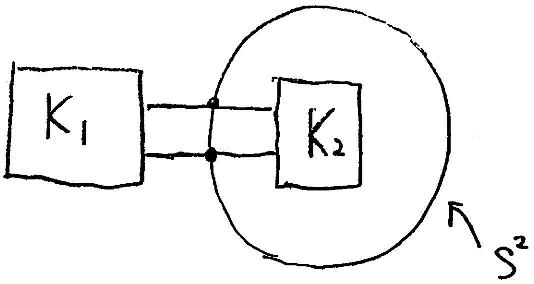
$K = K_1 \# K_2$.

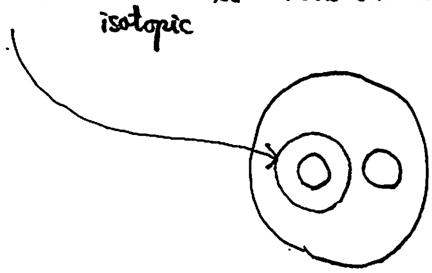


separating sphere

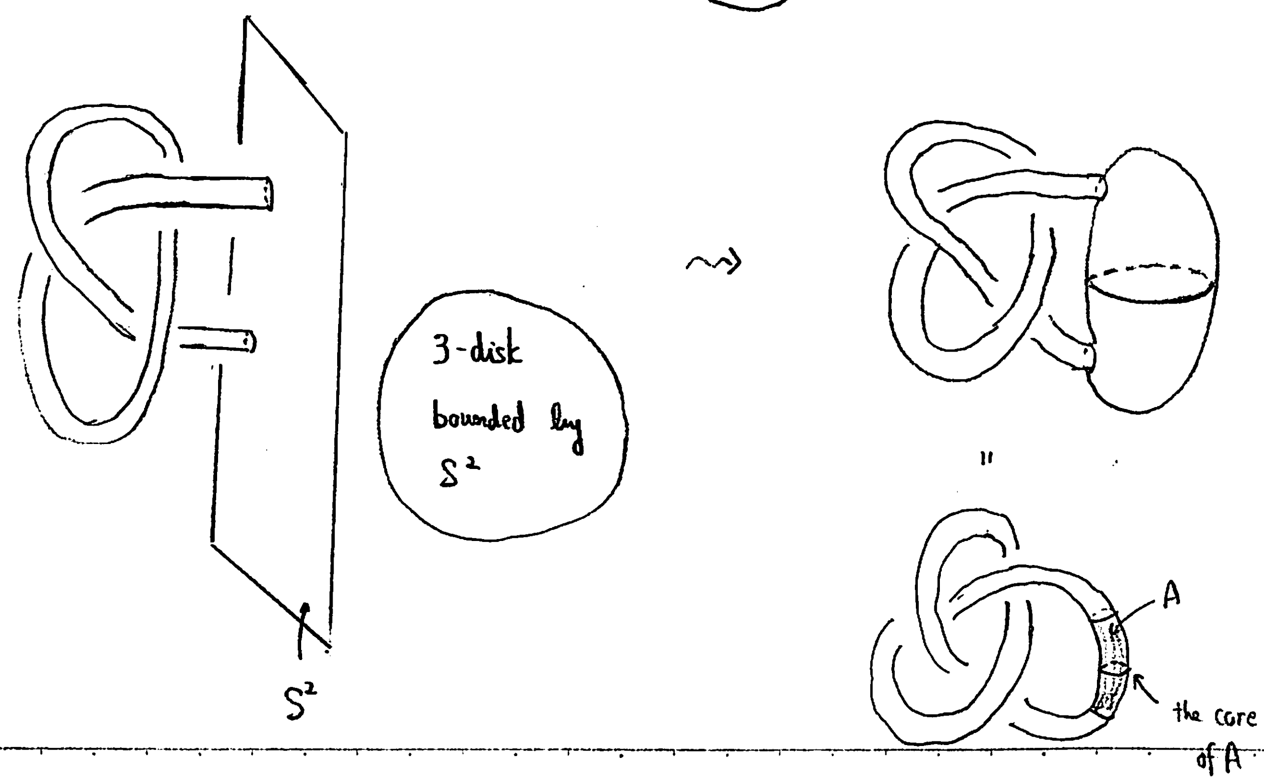
$N(K) \cap S^2$ = disjoint 2-disks.

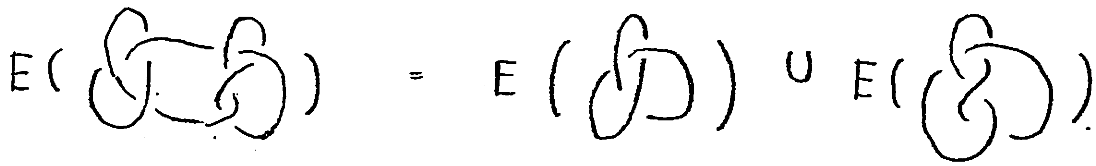
$A := E(K) \cap S^2 = S^2 \setminus (2 \text{ disks}) \cong S^1 \times I$.
annulus.

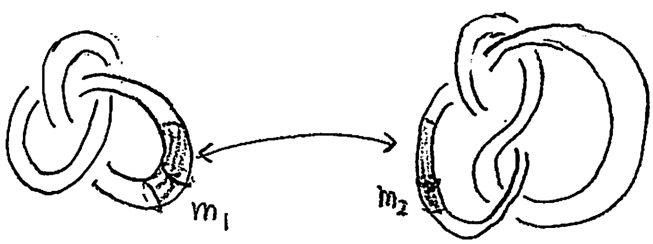
the core of A $\underset{\text{isotopic}}{\simeq}$ the meridian of K.



We examine knot exteriors







$$G(K_1 \# K_2) = G(K_1) \underset{\langle m_1\rangle = \langle m_2\rangle}{*} G(K_2)$$

$$= G(K_1) \underset{\langle m\rangle}{*} G(K_2)$$

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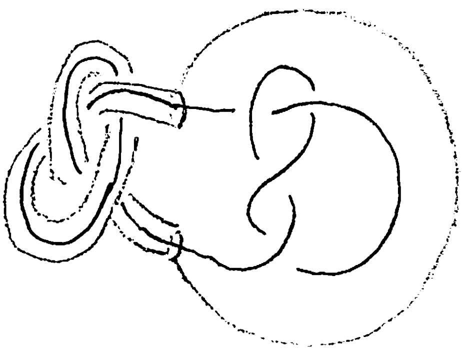
$E(K_1 \# K_2)$ contains 2 swallow - torus

$\cup$

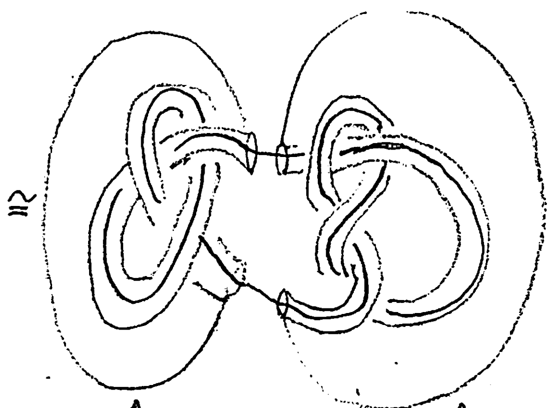
$E(K_i) \supset \partial E(K_i) = T_i^2$

$i = 1, 2$

We may choose $T_1 \cap T_2 = \emptyset$



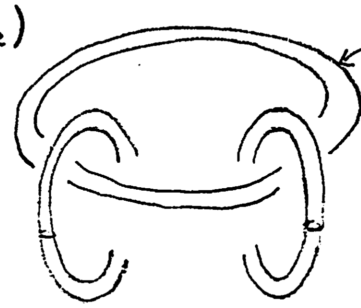
$\cong$



We choose T_i so that

$\mathrm{Int}\, E(K_1 \# K_2) \supset T_i \quad i = 1, 2.$ Then

$$E(K_1 \# K_2) = E(K_1) \underset{T_1}{\cup} Y \underset{T_2}{\cup} E(K_2)$$

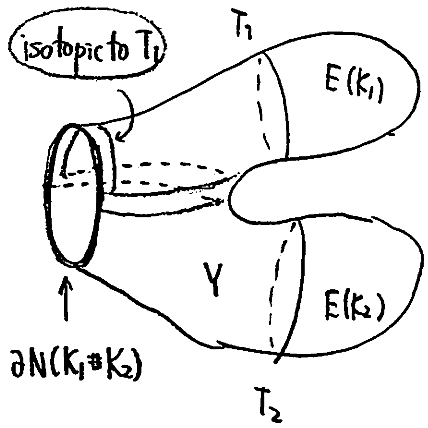
$Y \cong$  $\times \; S^1 \cong$ 

$\partial E(K_1 \# K_2)$ meridian

composing space.

Jaco - Shalen - Johanson decomposition

Remark $\quad Y = N(\partial E(K_1) \underset{A}{\cup} \partial E(K_2))$



⟨ Schematic picture ⟩

Remark. $\quad \pi_1(T_i) \to \pi_1(E(K_1 \# K_2))$ inj.

image $\cong \mathbb{Z}^2$

in $G(K_1 \# K_2)$.

Peripheral subgroup of knot group.

$$\underset{\substack{\parallel \\ T^2}}{\partial E(K)} \subset E(K).$$

Theorem (A consequence of Dehn's Lemma or Loop theorem)

$$K \neq 0 \iff j_* : \pi_1(\partial E(K)) \to \pi_1(E(K)) \text{ inj.}$$

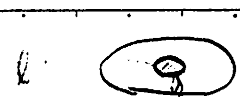
∴ The subgroup

$$\underset{\substack{\cong \\ \mathbb{Z} \oplus \mathbb{Z}}}{\operatorname{im} j_*} \leq G(K)$$

is called a <u>peripheral subgroup</u> of $G(K)$, denoted by $P(K)$.

Remark $\quad K = 0 \Rightarrow \operatorname{Ker}(\pi_1(\partial N(0)) \to \pi_1(E(0))) = \langle \ell \rangle \cong \mathbb{Z}$

(the longitude bounds a disk)



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Remark peripheral subgroup is defined up to conjugacy
(caused by the choice of base pt).

Theorem (Waldhausen: a consequence of a theorem on Haken manifolds).

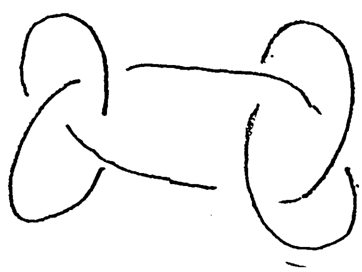
$$E(K_1) \cong E(K_2) \iff (G(K_1), P(K_1)) \cong (G(K_2), P(K_2))$$

i.e. $\exists\, f : G(K_1) \to G(K_2)$ an iso

s.t.

$$f(P(K_1)) = \gamma P(K_2) \gamma^{-1} \quad \exists\, \gamma \in G(K_2)$$

Example Isomorphic knot groups with different peripheral structures



granny knot

square knot.

granny knot $= K \# K$

square knot $= K \# K^*$

$$E(\text{granny}) = E(K) \underset{A}{\#} E(K)$$

$$E(\text{square}) = E(K) \underset{A}{\#} E(K^*)$$

$$= E(K) \underset{A'}{\#} E(K) \quad (*)$$

(*) Identified by the map
$$S^1 \times [-1,1] \to S^1 \times [-1,1]$$
$$id \times (-id_{[-1,1]}).$$

Hence $E(\text{granny}) \sim E(K) \#_{S^1} E(K) \sim E(\text{square})$.

$\therefore$ $G(\text{granny}) \cong G(\text{square})$.

Theorem Exotic homotopy equiv's between Haken manifold are induced.
i.e. h.e.'s not homotopic to any homeo.

by essential torus (Johanson).

$M \supset A$ annulus

$\check{M}$ = a manifold cutted along A
(with copies A_1, A_2 of A)

Define f by:
$$S^1 \times [-1,1] \xrightarrow{f} S^1 \times [-1,1]$$
$$(x, t) \mapsto (x, -t)$$
and attach A_1 and A_2 by f.

Then the resulting manifold $M' \simeq M$.

(but not $M' \not\cong M$. in general).

Exe $E(\text{granny}) \xrightarrow{f} E(\text{square})$ a homotopy equi.
$$\begin{matrix} E(\text{granny}) & & E(\text{square}) \\ \cup & & \cup \\ \partial E & & f(\partial E) \end{matrix} \quad ?$$

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Question

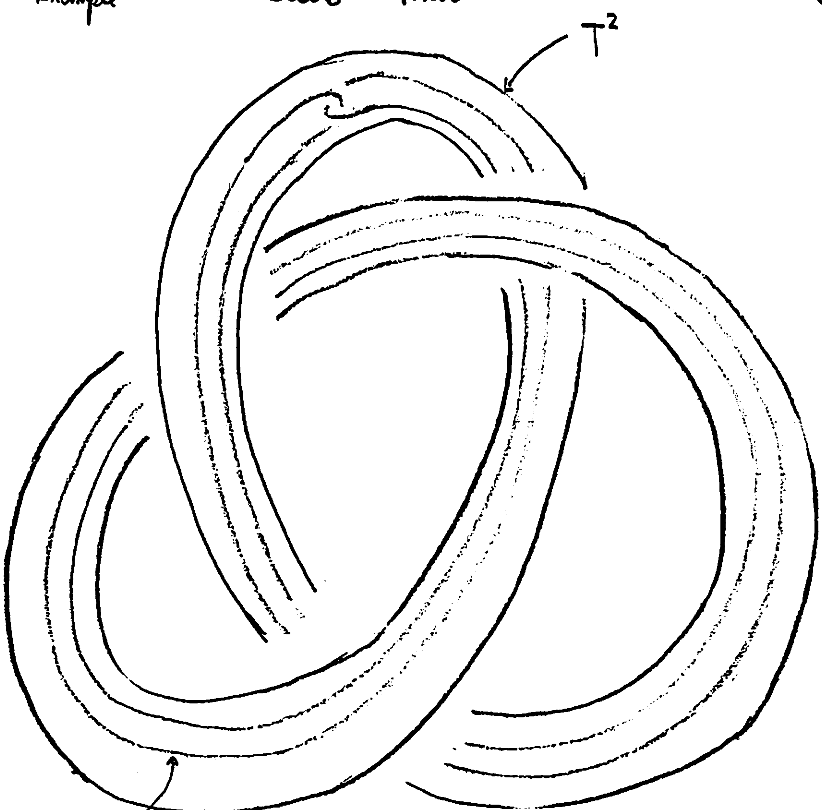
$H \leq G(K) \Rightarrow$ information on H ?

$H \cong \mathbb{Z}^2$

Example composite knot K

$G(K)$ contains a rank 2 abelian subgroup which is not peripheral.

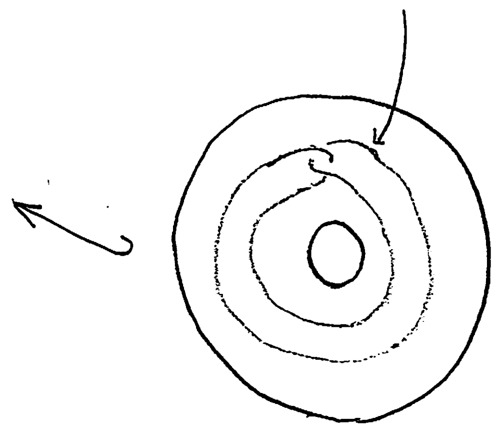
Example Satellite knot. (satellite knots) $\supset$ (composite knots)



an essential loop

i.e. (1) Not contained in a ball

(2) Not a core



K : a satellite knot.

$$\begin{array}{ccc} \pi_1(T^2) & \hookrightarrow & G(K) \\ \wr\Vert & & \uparrow \\ \mathbb{Z}^2 & & \text{not peripheral.} \end{array}$$

$K \subset V \subset S^3$

(i) V : core : nontrivial

$\pi_1(T) = \pi_1(\partial V) \to \pi_1(S^3 \setminus \mathring{V})$ inj

(ii) $K \subset V$ essential

(i) $K \not\subset B^3 \subset V$

(ii) $K \neq$ core

$\pi_1(V \setminus N(K)) \not\cong \mathbb{Z} \oplus \mathbb{Z}$

$G(K) = \pi_1(S^3 - V) *_{\pi_1(T)} \pi_1(V - N(K))$ //

Theorem $G(K)$ contains a non-peripheral rank 2 abelian group

$\Leftrightarrow$ (1) K : satellite

(2) K nontrivial torus knot

☺ $G(K) \supset Z(G(K)) = \langle f \rangle$.

$g \notin \langle f \rangle \Rightarrow \langle f, g \rangle \cong \mathbb{Z}^2$ nonperipheral.

Theorem. K : simple $\Leftrightarrow$ K non-torus, non-satellite

Theorem (Thurston)

K : simple $\Leftrightarrow$ K hyperbolic i.e. $S^3 \setminus K$ admits a complete hyperbolic str of finite volume.

JSJ decomposition of $E(K)$.

$E(K)$ contains a family of disjoint tori $\mathcal{T} = \{T_i\}$ s.t.

each piece obtained by cutting $E(K)$ by T_i is either

hyperbolic or Seifert fibered space

Seifert fibered piece is either

(1) torus knot exterior or

(2) composing space  $\times S^1$ or

(3) cable space

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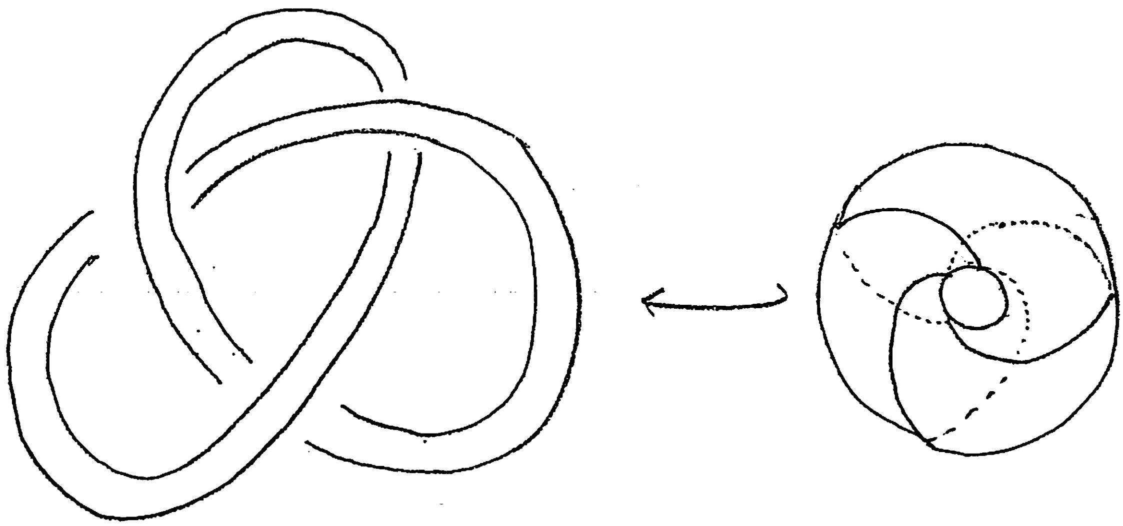
Cable knots.



< Schematic picture >