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Knot .

$$S^1 \hookrightarrow S^3 = \mathbb{R}^3 \cup \{\infty\} .$$

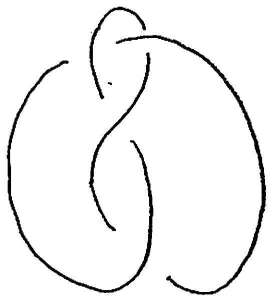

figure eight knot.

K_1, K_2 equivalent $\iff$ $(S^3, K_1) \cong (S^3, K_2)$

i.e.

$\exists\ f : S^3 \to S^3$ a homeo

s.t. $f(K_1) = K_2$.

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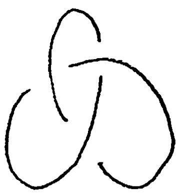
K_1, K_2 knots

K_1, K_2 equivalent $\underset{def}{\Longleftrightarrow}$ $(S^3, K_1) \cong (S^3, K_2)$

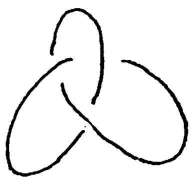
Remark 1. Orientation を考え.

$f : S^3 \to S^3$ orientation preserving homeo s.t.
$f(K_1) = K_2$ (強い意味の同値).

example



K



K^*

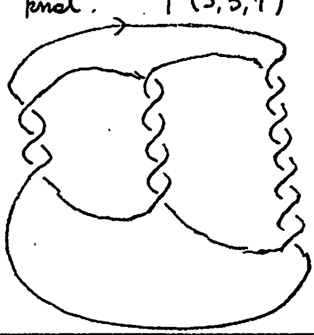
$$(S^3, K) \not\cong (-S^3, \mathrm{1}K) \,\, \| \,\, (S^3, \mathrm{1}K^*)$$

$\hat{r} : S^3 \to S^3$ reflection
$\hat{r}(K) = K^*$

Remark 2. K_1, K_2 orientation を考え.

$f : S^3 \circlearrowleft$ $\quad f| : K_1 \to K_2$ orientation pres. を考えることもある.

Pretzel knot. $P(3,5,7)$ は non-inv である. $\leftarrow$



$\sim$

$(S^3, K) \not\cong (S^3, -K)$
as oriented manifold pair

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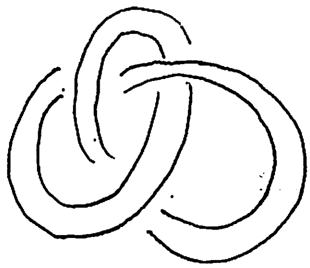
Krot complement. $X(K) = S^3 \setminus K$.

exterior $E(K) = S^3 \setminus \mathring{N}(K)$.

$\mathrm{Int}\, E(K) \approx X(K)$



$G(K) := \pi_1(E(K))$.

(Krot complementary theorem) $K_1 \cong K_2 \iff E(K_1) \cong E(K_2)$

Cameron - Luecke

$$\text{“}\iff\text{”} \quad G(K_1) \cong G(K_2)$$

$$(\Leftarrow \text{ if } K_1, K_2 \text{ prime })$$

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E(K) の geometric structure を理解する

Remark. (1) Fundamental groups play important roles in 3-manifold theory.

Poincaré conjecture

$$M: \text{a closed 3-manifold} \quad \pi_1(M) = 1$$
$$\Downarrow$$
$$M \approx S^3$$

(2) The same holds in 2-manifold theory

(3) The situation changes when the dim of manifold $\geqq 4$.

$\pi_1(M^4) = 1$ does not imply $M^4 = S^4$.
(in fact "many").

(4) (i) Characterizing knot groups is an open problem.

(ii) $\forall G$: finitely presented group

$\exists M^4$: an orientable closed 4-manifold s.t. $\pi_1(M^4) \cong G$.

(e.g. $\overset{n}{\#} S^3 \times S^1$ の surgery)

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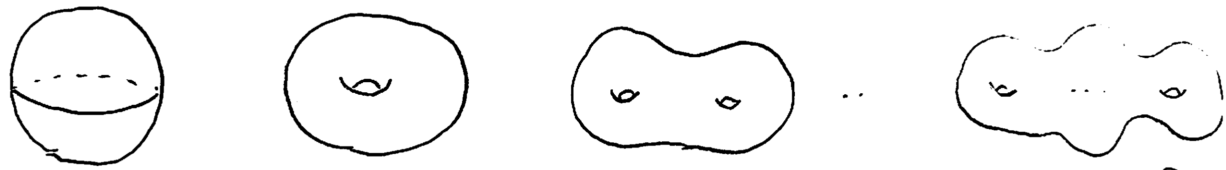
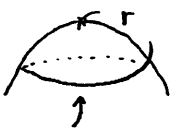
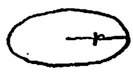
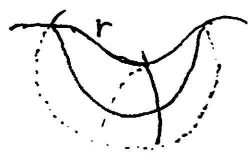
Warm up : (Surfaces)

closed orientable surface : classification



S_g

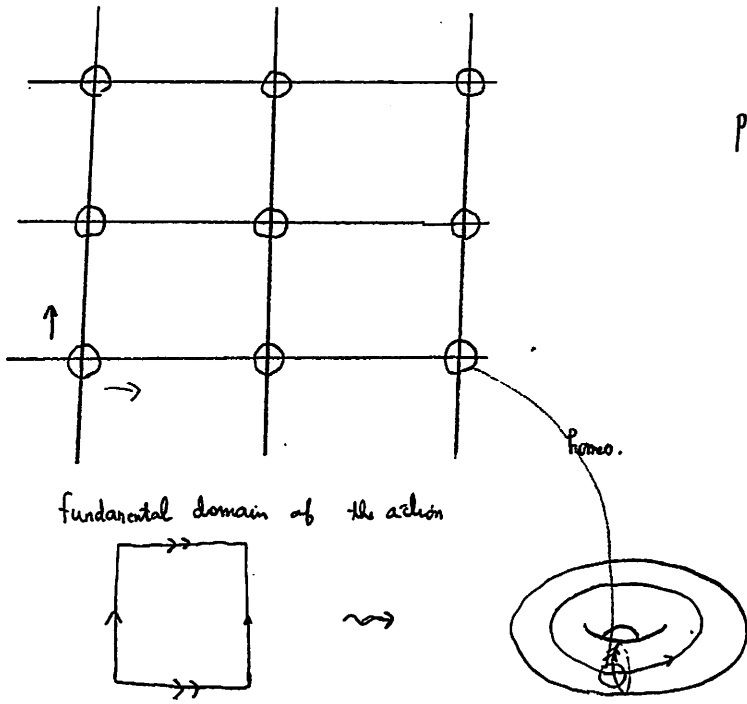
genus	0	1	2	g
χ	2	0	-2	$2-2g$
geom. str.	spherical geometry	Euclidian geom.	hyperbolic geometry	
local picture	 length $= 2\pi \sin r \approx 2\pi(r - \frac{1}{3!}r^3 \cdots)$	 $2\pi r$	 $2\pi \sinh r = 2\pi(r + \frac{1}{3!}r^3 + \cdots)$	
π_1	1	$\mathbb{Z} \oplus \mathbb{Z}$ abelian	$\langle a_1, b_1, \cdots, a_g, b_g \mid \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \rangle$ non abelian	

Covering spaces of torus

$T = S^1 \times S^1 = \mathbb{R}^2 / \mathbb{Z}^2$

$\mathbb{Z}^2 \curvearrowright \mathbb{R}^2$.

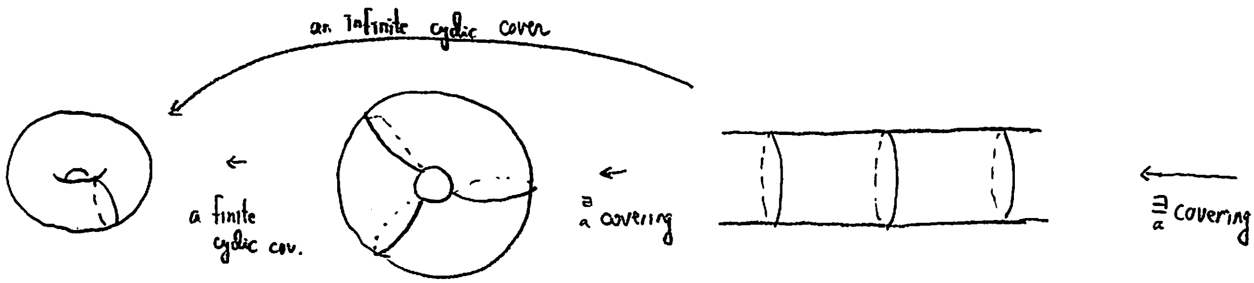
preserves the Euclidean structure



$p : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 = T$: the universal covering

simply conn.

the initial object of covering spaces.



Covering transformation group $\quad \mathrm{Aut}(\mathbb{R}^2, p) := $ the group of the covering transformations

$\cong \mathbb{Z}^2$ gen by $\rightarrow$ and $\uparrow$

$\cong \pi_1(T)$.

<u>Fact</u> top. space X (with a mild assumption) $\quad p : \tilde{X} \to X$ univ. covering

$$\mathrm{Aut}(\tilde{X}, p) \cong \pi_1(X).$$

(Isomorphism) $\quad b \in X$ base pt.

$\tilde{b} \in p^{-1}(b)$: fixed

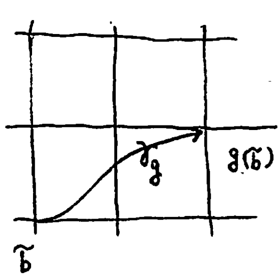
$g \in \mathrm{Aut}(\tilde{X}, p)$

$\tilde{\gamma}_g$: a path in $\tilde{X}$ from $\tilde{b}$ to $g(\tilde{b})$

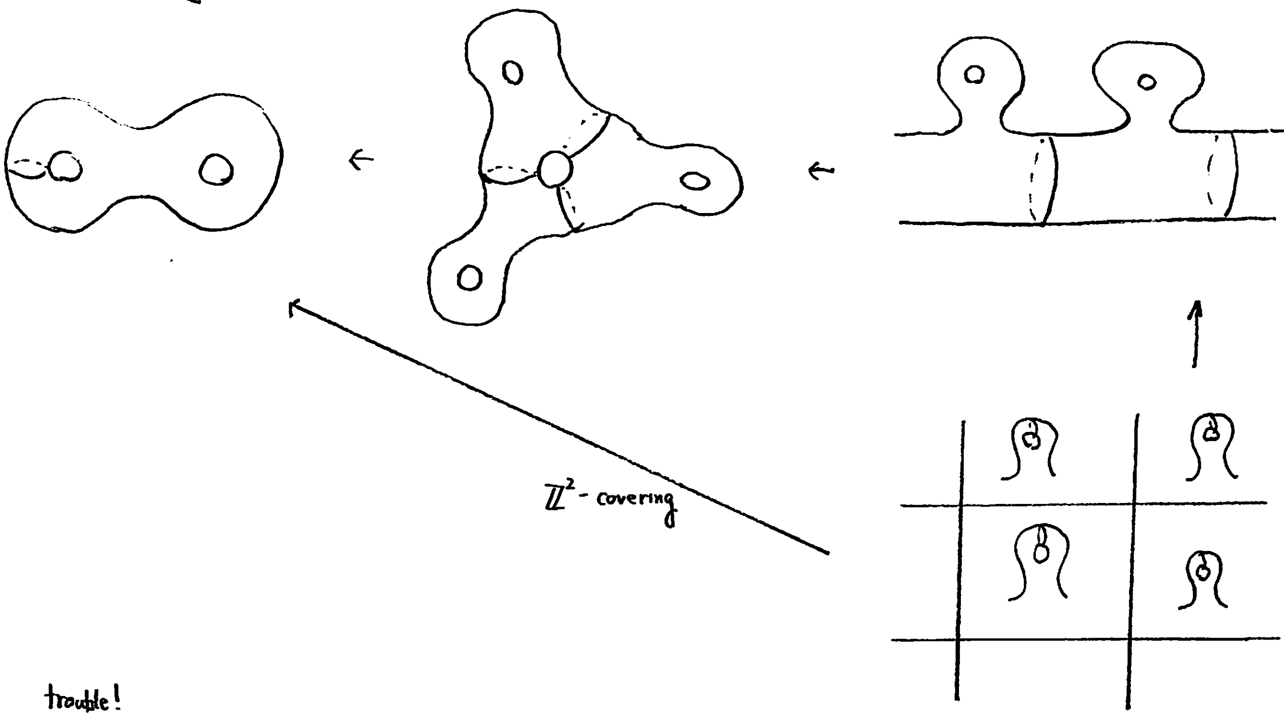
$\gamma_g := p \circ \tilde{\gamma}_g \in \pi_1(X, b)$.

$$\begin{array}{ccc} \mathrm{Aut}(\tilde{X}, p) & \to & \pi_1(X, b) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ g & \longmapsto & [\gamma_g] \end{array}$$

well-defined due to $\pi_1(\tilde{X}) = 1$.



Q. $\tilde{S}_2 = ?$



trouble!

$\pi_1(\tilde{S}_a)$: infinitely generated. $\subset \pi_1(S_2)$

hyperbolic geometry describes the universal covering $\tilde{S}_2$

Remark $\tilde{S}_2 \cong \mathbb{R}^2$.

$\pi_1(S_2)$ via van Kampen's theorem

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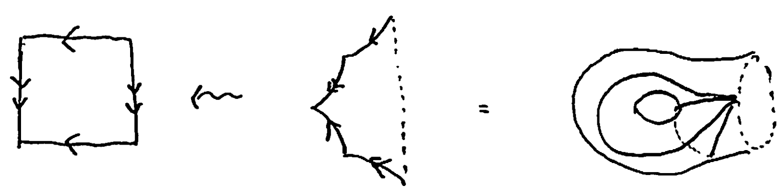
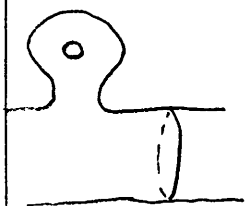




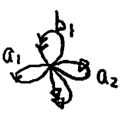
Van Kampen's theorem

$$\pi_1(S) \cong \pi_1(D) *_{\pi_1(S')} \pi_1(Y)$$

$\pi_1(S') = \langle \alpha \rangle \cong \mathbb{Z}$.

$\pi_1(Y) = \langle a_1, b_1, a_2, b_2 \mid - \rangle$

(∵)

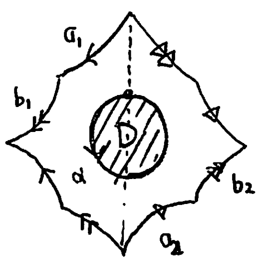
$Y \sim$ 

$\tilde{S}_a$

$$\begin{array}{ccccc} \pi_1(D) & \xleftarrow{i_1} & \pi_1(S') & \xrightarrow{i_2} & \pi_1(Y) \\ \parallel & & \cup & & \cup \\ 1 & \longleftarrow & \alpha & \longmapsto & a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \end{array}$$

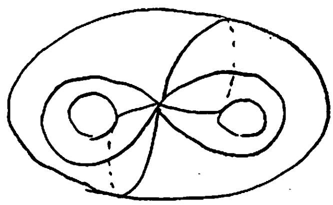
$[a_1,b_1][a_2,b_2] = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$

$\pi_1(S) = \pi_1(D) * \pi_1(Y) \Big/ i_1(\alpha) = i_2(\alpha)$

$P =$ 

$= D \cup_{S'} (P \setminus \mathring{D})$

$S_2 = P/\sim$

$=$ 

Exe Verify the above.

$= D \cup_{S'} (P - \mathring{D})/\sim$

$:= D \cup_{S'} Y$

$\pi_1(S_2) \cong \langle a_1, b_1, a_2, b_2 \mid [a_1,b_1][a_2,b_2] = 1 \rangle$

$= \langle a_1, b_1, a_2, b_2 \mid - \rangle \Big/$ Normal closure of $[a_1,b_1][a_2,b_2]$

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Exe. (1) Compute $\pi_1(S_g)$

(2) Compute $\pi_1(\overset{g}{\#} \mathbb{RP}^2)$

(3) Find a condition on the noncommutativity of $\pi_1(S_g)$ or $\pi_1(\overset{g}{\#} \mathbb{RP}^2)$

Exe. Verify that the connected sum is well-defined (of surfaces). What about higher dim'l manifolds?